Foundations of the New Field Theory.

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§ 1. *Introduction.*

The relation of matter and the electromagnetic field can be interpreted from two opposite standpoints :—

The first which may be called the *unitarian standpoint*‡ assumes only *one* physical entity, the electromagnetic field. The particles of matter are considered as singularities of the field and mass is a derived notion to be expressed by field energy (electromagnetic mass).

The second or *dualistic standpoint* takes field and particle as two essentially different agencies. The particles are the sources of the field, are acted on by the field but are not a part of the field ; their characteristic property is inertia measured by a specific constant, the mass.

At the present time nearly all physicists have adopted the dualistic view, which is supported by three facts.

1. *The failure of any attempt to develop a unitarian theory.*—Such attempts have been made with two essentially different tendencies : (*a*) The theories started by Heaviside, Searle and J. J. Thomson, and completed by Abraham, Lorentz, and others, make geometrical assumptions about the "shape" and kinematic behaviour of the electron and distribution of charge density (rigid electron of Abraham, contracting electron of Lorentz) ; they break down because they are compelled to introduce cohesive forces of non-electromagnetic origin ; (*b*) the theory of Mie§ formally avoids this difficulty by a generalization of Maxwell's equations making them non-linear ; this attempt breaks down

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‡ This expression has nothing to do with "unitary" field theory due to Einstein, Weyl, Eddington, and others where the problem consists of uniting the theories of gravitational and electro-magnetic fields into a kind of non-Riemannian geometry. Specially some of Eddington's formulæ, developed in § 101 of his book "The Mathematical Theory of Relativity" (Cambridge), have a remarkable formal analogy to those of this paper, in spite of the entirely different physical interpretation.

§ 'Ann. Physik,' vol. 37, p. 511 (1912) ; vol. 39, p. 1 (1912) ; vol. 40, p. 1 (1913). Also Born, 'Göttinger Nachr,' p. 23 (1914).

because Mie's field equations have the unacceptable property, that their solutions depend on the absolute value of the potentials.

2. *The result of relativity theory*, that the observed dependance of mass on velocity is in no way characteristic of electromagnetic mass, but can be derived from the transformation law.

3. Last, but not least, *the great success of quantum mechanics* which in its present form is essentially based on the dualistic view. It started from the consideration of oscillators and particles moving in a Coulomb field; the methods developed in these cases have then been applied even to the electromagnetic field, the Fourier coefficients of which behave like harmonic oscillators.

But there are indications that this quantum electrodynamics meets considerable difficulties and is quite insufficient to explain several facts.

The difficulties are chiefly connected with the fact that the self-energy of a point charge is infinite.† The facts unexplained concern the existence of elementary particles, the construction of the nuclei, the conversion of these particles into other particles or into photons, etc.

In all these cases there is sufficient evidence that the present theory (formulated by Dirac's wave equation) holds as long as the wave-lengths (of the Maxwell or of the de Broglie waves) are long compared with the "radius of the electron" e^2/mc^2, but breaks down for a field containing shorter waves. The non-appearance of Planck's constant in this expression for the radius indicates that in the first place the electromagnetic laws are to be modified; the quantum laws may then be adapted to the new field equations.

Considerations of this sort together with the conviction of the great philosophical superiority of the unitarian idea have led to the recent attempt‡ to construct a new electrodynamics, based on two rather different lines of thought: a new theory of the electromagnetic field and a new method of quantum mechanical treatment.

It seems desirable to keep these two lines separate in the further development. The purpose of this paper is to give a deeper foundation of the new field equations on classical lines, without touching the question of the quantum theory.

† The attempt to avoid this difficulty by a new definition of electric force acting on a particle in a given field, made by Wentzel ('Z. Physik,' vol. 86, pp. 479, 635 (1933), vol. 87, p. 726 (1934)), is very ingenious, but rather artificial and leads to new difficulties.

‡ Born, 'Nature,' vol. 132, p. 282 (1933); 'Proc. Roy. Soc.,' A, vol. 143, p. 410 (1934), cited here as I.

In the papers cited above, the new field theory has been introduced rather dogmatically, by assuming that the Lagrangian underlying Maxwell's theory

$$\mathsf{L} = \tfrac{1}{2}(\mathbf{H}^2 - \mathbf{E}^2) \tag{1.1}$$

(H and E are space-vectors of the electric and magnetic field) has to be replaced by the expression†

$$\mathsf{L} = b^2\left(\sqrt{1 + \frac{1}{b^2}(\mathbf{H}^2 - \mathbf{E}^2)} - 1\right). \tag{1.2}$$

The obvious physical idea of this modification is the following:—

The failure in the present theory may be expressed by the statement that it violates the *principle of finiteness* which postulates that a satisfactory theory should avoid letting physical quantities become infinite. Applying this principle to the velocity one is led to the assumption of an upper limit of velocity c and to replace the Newtonian action function $\frac{1}{2}mv^2$ of a free particle by the relativity expression $mc^2(1 - \sqrt{1 - v^2/c^2})$. Applying the same condition to the space itself one is lead to the idea of closed space as introduced by Einstein's cosmological theory.‡ Applying it to the electromagnetic field one is lead immediately to the assumption of an upper limit of the field strength and to the modification of the action function (1.1) into (1.2).

This argument seems to be quite convincing. But we believe that a deeper foundation of such an important law is necessary, just as in Einstein's mechanics the deeper foundation is provided by the postulate of relativity. Assuming that the expression $mc^2(1 - \sqrt{1 - v^2/c^2})$ has been found by the idea of a velocity limit it is seen that it can be written in the form

$$mc^2(1 - d\tau/dt),$$

where

$$c^2\,d\tau^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2,$$

and therefore it has the property that the time integral of $mc^2\,d\tau/dt$ is invariant for all transformations for which $d\tau^2$ is invariant. This four-dimensional group of transformations is larger than the three-dimensional group of transformations for which the time integral of the Newtonian function

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}m\,(ds/dt)^2\,; \qquad ds^2 = dx^2 + dy^2 + dz^2,$$

is invariant.

† See Born and Infeld, 'Nature,' vol. 132, p. 1004 (1933).

‡ See Eddington, "The Expanding Universe," Cambridge, 1933.

So we believe that we ought to search for a group of transformations for which the new Lagrangian expression has an invariant space-time integral and which is larger than that for the old expression (1.1). This latter group is the known group of special relativity but not the group of general space-time transformations.† Now it is very satisfying that the new Lagrangian belongs to this group of general relativity; we shall show that it can be derived from the postulate of general invariance with a few obvious additional assumptions. Therefore the new field theory seems to be a consequence of this very general principle, and the old one not more than a useful practical approximation, just in the same way as for the mechanics of Newton and Einstein.

In this paper we develop the whole theory from this general standpoint. We shall be obliged to repeat some of the formulæ published in the previous paper. The connection with the problems of gravitation and of quantum theory will be treated later.

§ 2. *Postulate of Invariant Action.*

We start from the general principle that all laws of nature have to be expressed by equations covariant for all space-time transformations. This, however, should not be taken to mean that the gravitational forces play an essential part in the constitution of the physical world; therefore we neglect the gravitational field so that there exist co-ordinate systems in which the metrical tensor g_{kl} has the value assumed in special relativity even in the centre of an electron. But we postulate that the natural laws are independent of the choice of the space-time co-ordinate system.

We denote space-time co-ordinates by

$$x^1, x^2, x^3, x^4 = x, y, z, ct.$$

The differential dx^k is, as usual, considered to be a contravariant vector. One can pull the indices up and down with help of the metrical tensor which in any cartesian co-ordinate system (as used in special relativity) has the form

$$(g_{kl}) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (\delta_{kl}). \qquad (2.1)$$

† The adaptation of the function L (1.1) to the general relativity by multiplication with $\sqrt{-g}$ is quite formal. Any expression can be made generally invariant in this way.

It is not the unit matrix, because of the different signs in the diagonal. Therefore we have to distinguish between covariant and contravariant tensors even in the co-ordinate systems of special relativity. In this case, however, the rule of pulling up and down of indices is very simple. This operation on the index 4 does not change the value of the tensor component, that on one of the indices 1, 2, 3 changes only the sign.

We use the well-known convention that one has to sum over any index which appears twice.

To obtain the laws of nature we use a variational principle of least action of the form

$$\delta \int \mathfrak{L}\, d\tau = 0, \qquad (d\tau = dx^1\, dx^2\, dx^3\, dx^4). \tag{2.2}$$

We postulate: the action integral has to be an invariant. We have to find the form of $\mathfrak{L}$ satisfying this condition.

We consider a covariant tensor field a_{kl}; we do not assume any symmetry property of a_{kl}. The question is to define $\mathfrak{L}$ to be such a function of a_{kl} that (2.2) is invariant. The well-known answer is that $\mathfrak{L}$ must have the form†

$$\mathfrak{L} = \sqrt{|a_{kl}|}; \qquad (|a_{kl}| = \text{determinant of } a_{kl}). \tag{2.3}$$

If the field is determined by several tensors of the second order, $\mathfrak{L}$ can be any homogeneous function of the determinants of the covariant tensors of the order $\frac{1}{2}$.

Each arbitrary tensor a_{kl} can be split up into a symmetrical and antisymmetrical part:

$$a_{kl} = g_{kl} + f_{kl}; \qquad g_{kl} = g_{lk}; \qquad f_{kl} = -f_{lk}. \tag{2.4}$$

The simplest simultaneous description of the metrical and electromagnetic field is the introduction of *one* arbitrary (unsymmetrical) tensor a_{kl}; we identify its symmetrical part g_{kl} with the metrical field, its antisymmetrical part with the electromagnetic field.‡

† See Eddington, "The Mathematical Theory of Relativity," Cambridge, § 48 and 101 (1923).

The proof is simple: by a transformation with the Jacobian $\mathrm{I} = \frac{\partial(\bar{x}^1 \ldots \bar{x}^4)}{\partial(x^1 \ldots x^4)}$ $d\tau$ is changed into $\overline{d\tau} = \mathrm{I}\, d\tau$ and $|a_{kl}|$ into $|\bar{a}_{kl}| = |a_{kl}|\, \mathrm{I}^{-2}$; for the dx^k are contravariant, a_{kl} covariant.

‡ This assumption has already been considered by Einstein, 'Berl. Ber.,' pp. 75/37 1923) and p. 414 (1925), from the standpoint of the affine field theory.

We have then three expressions which multiplied by $d\tau$ are invariant

$$\sqrt{-|a_{kl}|} = \sqrt{-|g_{kl}+f_{kl}|}\,; \qquad \sqrt{-|g_{kl}|}\,; \qquad \sqrt{|f_{kl}|}, \tag{2.5}$$

where the minus sign is added in order to get real values of the square roots; for (2.1) shows that $|\delta_{kl}| = -1$, therefore always $|g_{kl}| < 0$.

The simplest assumption for $\mathfrak{L}$ is any linear function of (2.5):

$$\mathfrak{L} = \sqrt{-|g_{kl}+f_{kl}|} + A\sqrt{-|g_{kl}|} + B\sqrt{|f_{kl}|}. \tag{2.6}$$

But the last term can be omitted. For if f_{kl} is the rotation of a potential vector, as we shall assume, its space-time integral can be changed into a surface integral and has no influence on the variational equation of the field.† Therefore we can take

$$B = 0. \tag{2.7}$$

We need another condition for the determination of A. Its choice is obvious. In the limiting case of the cartesian co-ordinate system and of small values of f_{kl}, $\mathfrak{L}$ has to give the classical expression

$$L = \tfrac{1}{4} f_{kl} f^{kl}. \tag{2.8}$$

We now leave the general co-ordinate system which has guided us to the expression (2.6) for $\mathfrak{L}$ and calculate $\mathfrak{L}$ in cartesian co-ordinates. Then we have with $g_{kl} = \delta_{kl}$ (see (2.1))

$$-|\delta_{kl}+f_{kl}| = -\begin{vmatrix} -1 & f_{12} & f_{13} & f_{14} \\ f_{21} & -1 & f_{23} & f_{24} \\ f_{31} & f_{32} & -1 & f_{34} \\ f_{41} & f_{42} & f_{43} & 1 \end{vmatrix} = 1 + (f_{23}^2 + f_{31}^2 + f_{12}^2 - f_{14}^2 - f_{24}^2 - f_{34}^2) - (f_{23}f_{14} + f_{31}f_{24} + f_{12}f_{34})^2$$

$$= 1 + (f_{23}^2 + f_{31}^2 + f_{12}^2 - f_{14}^2 - f_{24}^2 - f_{34}^2) - |f_{kl}|.$$

For small values of f_{kl} the last determinant can be neglected and (2.6) becomes equal to (2.8) only if

$$A = -1. \tag{2.9}$$

We have therefore the result:—

The action function of the electromagnetic field is in general co-ordinates

$$\mathfrak{L} = \sqrt{-|g_{kl}+f_{kl}|} - \sqrt{-|g_{kl}|}, \tag{2.10}$$

and in cartesian co-ordinates

$$L = \sqrt{1+F-G^2} - 1, \tag{2.11}$$

† See Eddington, *loc. cit.*, § 101.

where

$$F = f_{23}^2 + f_{31}^2 + f_{12}^2 - f_{14}^2 - f_{24}^2 - f_{34}^2 \tag{2.12}$$

$$G = f_{23}f_{14} + f_{31}f_{24} + f_{12}f_{34}. \tag{2.13}$$

Let us go back to the expression for $\mathfrak{L}$ in a general co-ordinate system. We denote as usual

$$|g_{kl}| = g,$$

and we develop the determinant $|g_{kl} + f_{kl}|$ into a power series in f_{kl}. We have then

$$|g_{kl} + f_{kl}| = g + \Phi(g_{kl}, f_{kl}) + |f_{kl}|.$$

The transformation properties of $|g_{kl} + f_{kl}|$, g, $|f_{kl}|$ and therefore also of $\Phi(g_{kl}, f_{kl})$ are the same. They transform in the same way as g. If we write

$$g + \Phi + |f_{kl}| = g\left(1 + \frac{\Phi}{g} + \frac{|f_{kl}|}{g}\right), \tag{2.14}$$

we see, that all expressions in the bracket on the right side of (2.14) are invariant. We have calculated their value in a geodetic co-ordinate system and have found:

$$\frac{\Phi}{g} = \tfrac{1}{2} f_{kl} f^{kl} = F = \tfrac{1}{2} f_{kr} f_{ls} g^{ik} g^{jr}.$$

Φ g is an invariant. We have therefore in an arbitrary co-ordinate system:

$$\left.\begin{aligned} |g_{kl} + f_{kl}| &= g(1 + F - G^2) \\ \mathfrak{L} &= \sqrt{-g}\,(\sqrt{1 + F - G^2} - 1) \end{aligned}\right\}, \tag{2.15}$$

$$F = \tfrac{1}{2} f_{kl} f^{kl}; \qquad G^2 = \frac{|f_{kl}|}{g} = \frac{(f_{23}f_{14} + f_{31}f_{24} + f_{12}f_{34})^2}{-g}. \tag{2.16}$$

Both F and G are invariant. We shall bring G into such a form, that its invariance will be evident. For this purpose let us define an antisymmetrical tensor j^{sklm} for any pair of indices, that is†

$$j^{sklm} \left\{\begin{aligned} &\frac{1}{2\sqrt{-g}} \text{ if } sklm \text{ is an even permutation of } 1, 2, 3, 4 \\ &\frac{-1}{2\sqrt{-g}} \text{ if } sklm \text{ is an odd permutation of } 1, 2, 3, 4 \\ &0 \text{ in any other case} \end{aligned}\right\}. \tag{2.17}$$

We can write now G in the following form:

$$G = \tfrac{1}{4} j^{sklm} f_{sk} f_{lm}. \tag{2.18}$$

† Einstein and Mayer, 'Berl. Ber.,' p. 3 (1932).

From the last equation we can deduce the tensor character of j^{sklm}. We can also write G in the form

$$G = \tfrac{1}{4} f_{sk} f^{*sk}, \tag{2.19}$$

where f^{*sk} is the *dual tensor* defined by

$$f^{*sk} = j^{sklm} f_{lm}, \tag{2.20}$$

that is

$$\left.\begin{aligned} f^{*23} &= \frac{1}{\sqrt{-g}} f_{14}, \quad f^{*31} = \frac{1}{\sqrt{-g}} f_{24}, \quad f^{*12} = \frac{1}{\sqrt{-g}} f_{34} \\ f^{*14} &= \frac{1}{\sqrt{-g}} f_{23}, \quad f^{*24} = \frac{1}{\sqrt{-g}} f_{31}, \quad f^{*34} = \frac{1}{\sqrt{-g}} f_{12} \end{aligned}\right\}, \tag{2.21}$$

or also

$$\left.\begin{aligned} f^*_{23} &= -\sqrt{-g}\, f^{14}, \quad f^*_{31} = -\sqrt{-g} f^{24}, \quad f^*_{12} = -\sqrt{-g} f^{34} \\ f^*_{14} &= -\sqrt{-g}\, f^{23}, \quad f^*_{24} = -\sqrt{-g} f^{31}, \quad f^*_{34} = -\sqrt{-g} f^{12} \end{aligned}\right\}, \tag{2.22}$$

because

$$f^*_{sk} = j_{sklm} f^{lm}; \qquad j_{sklm} = g j^{sklm} = g_{as} g_{bk} g_{cl} g_{dm} j^{abcd}. \tag{2.23}$$

We shall need later the following formulæ:

$$f^{*kl} f^*_{kl} = -f^{kl} f_{kl} \tag{2.24}$$

$$j^{lsab} f_{ks} f_{ab} = f^{*ls} f_{ks} = G \delta_k^{\ l} \tag{2.25}$$

$$f^{**ls} = -f^{ls}. \tag{2.26}$$

(2.24)–(2.26) follow from the definition of f^*_{kl}, f^{*kl} and G given above.

The function $\mathfrak{L}$ represented by (2.15) is the simplest Lagrangian satisfying the principle of general invariance. But it differs from that considered in I by the term G^2. This is of the fourth order in the f_{kl} and can, therefore, be neglected except in the immediate neighbourhood of singularities (*i.e.*, electrons, see § 6). But the Lagrangian used in I can also be expressed in a general covariant form; for G^2 is a determinant, namely, $|f_{kl}|$, therefore

$$\int (\sqrt{-|g_{kl} + f_{kl}| + |f_{kl}|} - \sqrt{-|g_{kl}|})\, d\tau \tag{2.27}$$

is also invariant; in cartesian co-ordinates it has exactly the form

$$\int (\sqrt{1 + \mathrm{F}} - 1)\, d\tau. \tag{2.28}$$

Which of these action principles is the right one can only be decided by their consequences. We take the expression given by (2.15) and can then easily

return to the other (2.27) or (2.28) by putting $G = 0$. In any case the solution of the *statical* problem is identical for both action functions because one has $G = 0$ in this special case.

§ 3. *Action Principle, Field Equation and Conservation Law.*

We write (2.15) in the general form

$$\mathfrak{L} = \sqrt{-g}\,\mathsf{L} = \sqrt{-g}\,\mathsf{L}\,(g_{kl}, F, G).$$

We shall see that all considerations hold if L is an invariant function of these arguments. As usual we assume the existence of a potential vector ϕ_k, so that

$$f_{kl} = \frac{\partial \phi_l}{\partial x^k} - \frac{\partial \phi_k}{\partial x^l}. \tag{3.1}$$

Then we have the identity

$$\frac{\partial f_{lm}}{\partial x^k} + \frac{\partial f_{mk}}{\partial x^l} + \frac{\partial f_{kl}}{\partial x^m} = 0, \tag{3.2}$$

which can with the help of (2.20) be written:

$$\frac{\partial \sqrt{-g} f^{*kl}}{\partial x^l} = 0. \tag{3.2A}$$

We introduce a second kind of antisymmetrical field tensor p_{kl}, which has to f_{kl} a relation similar to that which, in Maxwell's theory of macrospic bodies, the dielectric displacement and magnetic induction have to the field strengths:

$$\sqrt{-g}\,p^{kl} = \frac{\partial \mathfrak{L}}{\partial f_{kl}} = \sqrt{-g}\left(2\frac{\partial \mathsf{L}}{\partial F} f^{kl} + \frac{\partial \mathsf{L}}{\partial G} f^{*kl}\right) = \frac{(f^{kl} - G f^{*kl})\sqrt{-g}}{\sqrt{1 + F - G^2}}. \tag{3.3}$$

The variation principle (2.2) gives the Eulerian equations

$$\frac{\partial \sqrt{-g}\,p^{kl}}{\partial x^l} = 0. \tag{3.4}$$

The equation (3.2) (or (3.2A)) and (3.4) are the complete set of field equations.

We prove the validity of the conservation law as in Maxwell's theory. Assuming a geodetic co-ordinate system, we multiply (3.2) by p^{lm}:

$$p^{lm}\left(\frac{\partial f_{lm}}{\partial x^k} + \frac{\partial f_{mk}}{\partial x^l} + \frac{\partial f_{kl}}{\partial x^m}\right) = 0. \tag{3.5}$$

In the second and third term we can take p^{lm} under the differentiation symbol because of (3.4); in the first term we use the definition (3.3) of p^{lm}:

$$2\frac{\partial}{\partial x^l}(p^{lm}f_{mk}) + \frac{\partial \mathsf{L}}{\partial f_{lm}}\frac{\partial f_{lm}}{\partial x^k} = 0,$$

or

$$-2\frac{\partial}{\partial x^l}(p^{ml}f_{mk}) + 2\frac{\partial \mathsf{L}}{\partial x^k} = 0.$$

If we introduce the tensor

$$\mathrm{T}_k^{\ l} = \mathsf{L}\delta_k^{\ l} - p^{ml}f_{mk}, \tag{3.6}$$

where

$$\delta_k^{\ l} = \left\{\begin{matrix} 1 \text{ if } k = l \\ 0 \text{ if } k \neq l \end{matrix}\right\}, \tag{3.7}$$

we have

$$\frac{\partial \mathrm{T}_k^{\ l}}{\partial x^l} = 0. \tag{3.8}$$

In an arbitrary co-ordinate system we have

$$\frac{\partial \sqrt{-g}\,\mathrm{T}_k^{\ l}}{\partial x^l} - \tfrac{1}{2}\sqrt{-g}\mathrm{T}^{ab}\frac{\partial g_{ab}}{\partial x^k} = 0, \tag{3.9}$$

or, with the usual notation of covariant differentiation

$$\mathrm{T}^l_{\ k\,l} = 0. \tag{3.9A}$$

It follows from (3.3) and (2.25) that we can write also $\mathrm{T}_k^{\ l}$ in the form

$$\mathrm{T}_k^{\ l} = \mathsf{L}\,\delta_k^{\ l} - \frac{f^{ml}f_{mk} - \mathrm{G}^2\delta_k^{\ l}}{\sqrt{1 + \mathrm{F} - \mathrm{G}^2}}. \tag{3.6A}$$

§ 4. *Lagrangian and Hamiltonian.*

L can be considered as a function of g^{kl} and f_{kl}. We shall show that $\frac{-2}{\sqrt{-g}}\frac{\partial \mathsf{L}}{\partial g^{kl}}$ is the energy-impulse tensor. We find

$$\frac{\partial \sqrt{-g}}{\partial g^{kl}} = -\tfrac{1}{2}\sqrt{-g}\,g_{kl} \tag{4.1}$$

$$\frac{\partial \mathrm{F}}{\partial g^{kl}} = g^{sr}f_{ks}f_{lr} \tag{4.2}$$

$$\frac{\partial \mathrm{G}}{\partial g^{kl}} = \tfrac{1}{2}\mathrm{G}g_{kl}. \tag{4.3}$$

Therefore

$$-2\frac{\partial \mathscr{L}}{\partial g^{kl}} = \sqrt{-g}\left\{\mathsf{L}g_{kl} - 2\left(\frac{\partial \mathsf{L}}{\partial \mathrm{F}}\frac{\partial \mathrm{F}}{\partial g^{kl}} + \frac{\partial \mathsf{L}}{\partial \mathrm{G}}\frac{\partial \mathrm{G}}{\partial g^{kl}}\right)\right\}$$

$$= \sqrt{-g}\left\{\mathsf{L}g_{kl} - \frac{f_{ks}f_{lr}g^{sr} - \mathrm{G}^2 g_{kl}}{\sqrt{1+\mathrm{F}-\mathrm{G}^2}}\right\}. \tag{4.4}$$

It follows from (3.6A) and (3.6)

$$-2\frac{\partial \mathscr{L}}{\partial g^{kl}} = \sqrt{-g}\,\mathrm{T}_{kl} = \sqrt{-g}\,(\mathsf{L}g_{kl} - f_{ks}p_{lr}g^{sr}). \tag{4.5}$$

Now it is very easy to generalize our action principle in such a way that it contains Einstein's gravitation laws; one has only to add to the action integral the term $\int \mathrm{R}\sqrt{-g}\,d\tau$, where R is the scalar of curvature. But we do not discuss problems connected with gravitation in this paper.

$\mathscr{L}$ was regarded as a function of g^{kl} and f_{kl}. We can, however, express $\mathscr{L}$ also as a function of g^{kl} and p_{kl}. It can be shown that it is possible to solve the equations

$$p^{kl} = \frac{f^{kl} - \mathrm{G}f^{*kl}}{\sqrt{1+\mathrm{F}-\mathrm{G}^2}} \tag{3.3}$$

with respect to f^{kl}. For this purpose we have to calculate

$$\tfrac{1}{2}p^{*kl}p^{*}{}_{kl} = \mathrm{P}, \tag{4.6}$$

$$\tfrac{1}{4}p^{kl}p^{*}{}_{kl} = \mathrm{Q}, \tag{4.7}$$

i.e., P and Q corresponding to F and G. Using the formulæ (3.3) and (2.24)–(2.26), we obtain

$$\mathrm{P} = \frac{-\mathrm{F}+\mathrm{G}^2\mathrm{F}+4\mathrm{G}^2}{1+\mathrm{F}-\mathrm{G}^2} \tag{4.8}$$

$$\mathrm{Q} = \mathrm{G}. \tag{4.9}$$

The last equations can be written in a more symmetrical form:

$$\frac{1+\mathrm{F}-\mathrm{G}^2}{1+\mathrm{G}^2} = \frac{1+\mathrm{Q}^2}{1+\mathrm{P}-\mathrm{Q}^2} \tag{4.8A}$$

$$\mathrm{G} = \mathrm{Q}. \tag{4.9A}$$

We are now able to solve the equations (3.3). It follows from (3.3) and (2.26) that

$$p^{*kl} = \frac{f^{*kl} + \mathrm{G}f^{kl}}{\sqrt{1+\mathrm{F}-\mathrm{G}^2}}. \tag{3.3A}$$

Solving (3.3) and (3.3A) we obtain (taking into account (4.8A) and (4.9A))

$$f^{kl} = \frac{p^{kl} + Qp^{*kl}}{\sqrt{1 + P - Q^2}}; \qquad f^{*kl} = \frac{p^{*kl} - Qp^{kl}}{\sqrt{1 + P - Q^2}}. \tag{4.10}$$

The tensors f_{kl} and p_{kl} can now be treated completely symmetrically. Instead of the Lagrangian L we can use in the principle of action the Hamiltonian function H:

$$\mathsf{H} = \mathsf{L} - \tfrac{1}{2} p^{kl} f_{kl}, \tag{4.11}$$

where H has to be regarded as a function of g^{kl} and p_{kl}. From (4.8), (4.9), and (4.10) it follows for H as a function of g^{kl} and p_{kl}:

$$\mathfrak{H} = \mathsf{H}\sqrt{-g} = \sqrt{-g}\,(\sqrt{1 + P - Q^2} - 1), \tag{4.12}$$

and this can be expressed in the form

$$\mathfrak{H} = \sqrt{-|g_{kl} + p^*_{kl}|} - \sqrt{-|g_{kl}|}. \tag{4.12A}$$

The function H leads us to exactly the same equations of the field as the function L. We see that the equations

$$p^*_{kl} = \frac{\partial \psi^*_l}{\partial x^k} - \frac{\partial \psi^*_k}{\partial x^l} \qquad (\psi^*_k = \text{anti-potential-vector}) \tag{4.13}$$

$$\sqrt{-g} f^{*kl} = \frac{\partial \mathfrak{H}}{\partial p^*_{kl}} \tag{4.14}$$

$$\frac{\partial \sqrt{-g} f^{*kl}}{\partial x^l} = 0. \tag{4.15}$$

are entirely equivalent to the equations (3.4), (4.10), (3.2A).

The energy-impulse tensor (3.6A) can also be expressed with help of H instead of L. One has

$$T_k{}^l = \mathsf{H}\delta_k{}^l - f^{*ml} p^*_{mk} = (\mathsf{L} - \tfrac{1}{2} p^{ab} f_{ab})\,\delta_k{}^l - f^{*ml} p^*_{mk}. \tag{4.16}$$

The identity of this expression with (3.6) is evident,† if we appeal to the following formula which can be deduced from (2.21), (2.22)

$$f^{*ml} p^*_{mk} = p^{ml} f_{mk} - \tfrac{1}{2} p^{ab} f_{ab}\,\delta_k{}^l \tag{4.17}$$

† In I it has been stated that the two expressions for $T_k{}^l$, obtained with help of L and H, are different; this has turned out to be a mistake.

Generally, from each equation containing $\mathfrak{L}$, ϕ_k, f_{kl}, $-p_{kl}$, one obtains another correct equation changing these quantities correspondingly into $\mathfrak{H}$, ψ^*_k, p^*_{kl}, f^*_{kl}.

§ 5. *Field Equations in Space-vector Form.*

We now introduce the conventional units instead of the natural units. We denote by **B**, **E** and **D**, **E**, the space-vectors which characterize the electromagnetic field in the conventional units. We have in a cartesian co-ordinate system :

$$(x^1, x^2, x^3, x^4) \rightarrow (x, y, z, ct) \tag{5.1}$$

$$(\phi_1, \phi_2, \phi_3, \phi_4) \rightarrow (\mathbf{A}, \phi) \tag{5.2}$$

$$\left.\begin{aligned}(f_{23}, f_{31}, f_{12} &\rightarrow \mathbf{B}\\ f_{14}, f_{24}, f_{34}) &\rightarrow \mathbf{E}\end{aligned}\right\} \tag{5.3}$$

$$\left.\begin{aligned}(p_{23}, p_{31}, p_{12} &\rightarrow \mathbf{H}\\ p_{14}, p_{24}, p_{34}) &\rightarrow \mathbf{D}\end{aligned}\right\} \tag{5.4}$$

The quotient of the field strength expressed in the conventional units divided by the field strength in the natural units may be denoted by b. This constant of a dimension of a field strength may be called the *absolute field* ; later we shall determine the value of b, which turns out to be very great, *i.e.*, of the order of magnitude 10^{16} e.s.u.

We have

$$\mathsf{L} = \sqrt{1 + \mathrm{F} - \mathrm{G}^2} - 1, \tag{2.11}$$

$$\mathrm{F} = \frac{1}{b^2}(\mathbf{B}^2 - \mathbf{E}^2) : \qquad \mathrm{G} = \frac{1}{b^2}(\mathbf{B} \cdot \mathbf{E}) \tag{2.12A), (2.13A}$$

$$\left.\begin{aligned}\mathbf{H} = b^2 \frac{\partial \mathsf{L}}{\partial \mathbf{B}} &= \frac{\mathbf{B} - \mathrm{G}\mathbf{E}}{\sqrt{1 + \mathrm{F} - \mathrm{G}^2}}\\ \mathbf{D} = b^2 \frac{\partial \mathsf{L}}{\partial \mathbf{E}} &= \frac{\mathbf{E} - \mathrm{G}\mathbf{B}}{\sqrt{1 + \mathrm{F} - \mathrm{G}^2}}.\end{aligned}\right\} \tag{3.3A}$$

$$\mathbf{B} = \mathrm{rot}\,\mathbf{A}\,; \qquad \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\,\phi \tag{3.1A}$$

$$\mathrm{rot}\,\mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0\,; \qquad \mathrm{div}\,\mathbf{B} = 0 \tag{3.2B}$$

$$\mathrm{rot}\,\mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} = 0\,; \qquad \mathrm{div}\,\mathbf{D} = 0. \tag{3.4A}$$

Our field equations (3.2B) and (3.4A) are formally identical with Maxwell's equations for a substance which has a dielectric constance and a susceptibility,

being certain functions of the field strength, but without a spatial distribution of charge and current.

For the energy-impulse tensor we find :

$$\left(\frac{1}{4\pi}T^{kl}\right) = \begin{pmatrix} X_x & X_y & X_z & cG_x \\ Y_x & Y_y & Y_z & cG_y \\ Z_x & Z_y & Z_z & cG_z \\ \frac{1}{c}S_x & \frac{1}{c}S_y & \frac{1}{c}S_z & U \end{pmatrix} \tag{3.6A}$$

$$\left.\begin{aligned} 4\pi X_x &= H_y B_y + H_z B_z - D_x E_x - b^2 \mathsf{L} \\ 4\pi Y_x &= 4\pi X_y = -H_y B_x - D_x E_y \\ \frac{4\pi}{c}S_x &= 4\pi c G_x = D_y B_z - D_z B_y \\ 4\pi U &= D_x E_x + D_y E_y + D_z E_z + b^2 \mathsf{L} \end{aligned}\right\} \tag{3.6B}$$

One gets another set of expressions for these quantities by changing L, $\mathbf{B}$, $\mathbf{E}$, $\mathbf{H}$, $\mathbf{D}$ into H, $\mathbf{H}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{E}$.

The conservation laws are :

$$\left.\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} &= -\frac{1}{c^2}\frac{\partial S_x}{\partial t} \\ \frac{\partial S_x}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial S_z}{\partial z} &= -\frac{\partial U}{\partial t} \end{aligned}\right\} \tag{3.8A}$$

The function H is given by

$$\mathsf{H} = \sqrt{1 + \mathrm{P} - \mathrm{Q}^2} - 1 \tag{4.12A}$$

$$\mathrm{P} = \frac{1}{b^2}(\mathbf{D}^2 - \mathbf{H}^2); \qquad \mathrm{Q} = \frac{1}{b^2}(\mathbf{D}\,.\,\mathbf{H}). \tag{4.6A}; \ (4.7A)$$

Solving (3.3A) we obtain :

$$\left.\begin{aligned} \mathbf{B} = b^2\frac{\partial \mathsf{H}}{\partial \mathbf{H}} &= \frac{\mathbf{H} + \mathrm{Q}\mathbf{D}}{\sqrt{1 + \mathrm{P} - \mathrm{Q}^2}} \\ \mathbf{E} = b^2\frac{\partial \mathsf{H}}{\partial \mathbf{D}} &= \frac{\mathbf{D} + \mathrm{Q}\mathbf{H}}{\sqrt{1 + \mathrm{P} - \mathrm{Q}^2}} \end{aligned}\right\} \tag{3.10A}$$

§ 6. *Static Solution of the Field Equations.*

We consider (in the cartesian co-ordinate system) the electrostatic case where $\mathbf{B} = \mathbf{H} = 0$ and all other field components are independent of t. Then the field equations reduce to :

$$\mathrm{rot}\,\mathbf{E} = 0 \tag{6.1}$$

$$\mathrm{div}\,\mathbf{D} = 0. \tag{6.2}$$

We solve this equation for the case of central symmetry. Then (6.2) is simply

$$\frac{d}{dr}(r^2 D_r) = 0, \tag{6.3}$$

and (6.3) has the solution

$$D_r = e/r^2. \tag{6.4}$$

In this case the field **D** is exactly the same as in Maxwell's theory: the sources of **D** are point charges given by the surface integral

$$4\pi e = \int D_r \, d\sigma. \tag{6.5}$$

The equation (6.1) gives

$$E_r = -\frac{d\phi}{dr} = -\phi'(r) \tag{6.6}$$

and from (3.3A)

$$D_r = \frac{E_r}{\sqrt{1 - \frac{1}{b^2} E_r^2}} = -\frac{\phi'(r)}{\sqrt{1 - \frac{1}{b^2}\phi'^2}}. \tag{6.7}$$

The combination of (6.4) and (6.7) gives a differential equation for $\phi(r)$ of the first order, with the solution

$$\phi(r) = \frac{e}{r_0} f\left(\frac{r}{r_0}\right); \; f(x) = \int_x^\infty \frac{dy}{\sqrt{1 + y^4}}; \; r_0 = \sqrt{\frac{e}{b}}. \tag{6.8}$$

This is the elementary potential of a point charge e, which has to replace Coulomb's law; the latter is an approximation for $x \gg 1$, as is seen immediately, but the new potential is finite everywhere.

With help of the substitution $x = \tan \frac{1}{2}\beta$ one obtains

$$f(x) = \tfrac{1}{2} \int_{\bar{\beta}(x)}^{\pi} \frac{d\beta}{\sqrt{1 - \frac{1}{2}\sin^2\beta}} = f(0) - \tfrac{1}{2} F\left(\frac{1}{\sqrt{2}}, \bar{\beta}\right), \tag{6.9}$$

where

$$\bar{\beta} = 2 \arctan x, \tag{6.10}$$

and $F(k,\beta)$ is the Jacobian elliptic integral of the first kind for $k = \frac{1}{\sqrt{2}} = \sin \frac{1}{4}\pi$ (tabulated in many books)†

$$F\left(\frac{1}{\sqrt{2}}, \bar{\beta}\right) = \int_0^{\bar{\beta}} \frac{d\beta}{\sqrt{1 - \frac{1}{2}\sin^2\beta}}. \tag{6.11}$$

† *E.g.*, Jahnke-Emde, "Tables of functions" (Teubner 1933), p. 127.

For $x = 0$ one has

$$f(0) = \mathrm{F}\left(\frac{1}{\sqrt{2}}, \tfrac{1}{2}\pi\right) = 1{,}8541. \tag{6.12}$$

The potential has its maximum in the centre and its value is

$$\phi(0) = 1{,}8541\, e/r_0. \tag{6.13}$$

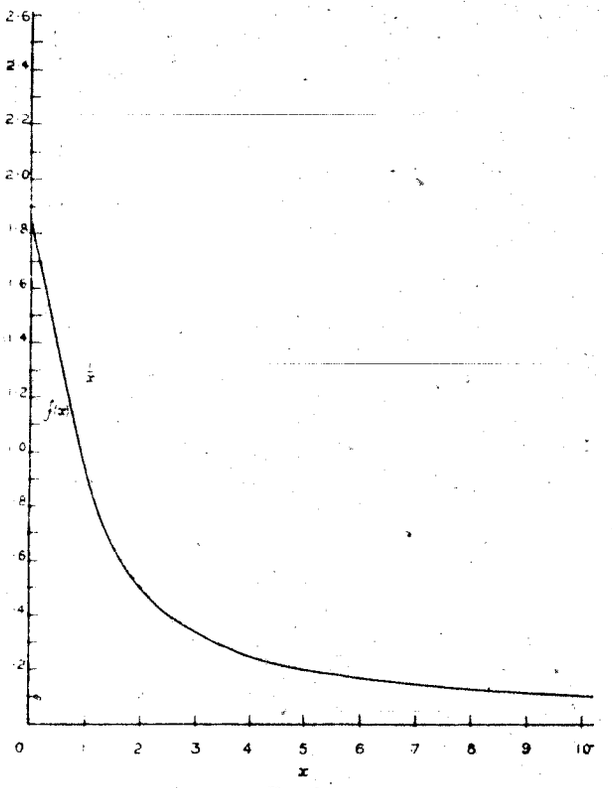

FIG. 1.

The function $f(x)$ is plotted in fig. 1. It has very similar properties to the function arc cot x. For example, one has

$$\bar{\beta}(1/x) = 2 \arctan 1/x = 2(\tfrac{1}{2}\pi - \arctan x) = \pi - \bar{\beta}(x);$$

on the other hand

$$\mathrm{F}\left(\frac{1}{\sqrt{2}}, \bar{\beta}\right) + \mathrm{F}\left(\frac{1}{\sqrt{2}}, \pi - \bar{\beta}\right) = \mathrm{F}\left(\frac{1}{\sqrt{2}}, \pi\right).$$

Therefore one has

$$f(x) + f(1/x) = f(0). \tag{6.14}$$

It is sufficient, therefore, to calculate $f(x)$ from $x = 0$ to $x = 1$ or from $\beta = 0$ to $\beta = \frac{1}{4}\pi$.

One sees, that the **D** field is infinite for $r = 0$; **E** and ϕ, however, are always finite. One has

$$\mathrm{D}_r = e/r^2, \tag{6.4}$$

$$\mathrm{E}_r = \frac{e}{r_0^2\sqrt{1 + (r/r_0)^4}}. \tag{6.15}$$

The components E_x, E_y, E_z are finite at the centre, but have there a discontinuity.

§ 7. *Sources of the Field.*

In the older theories, which we have called dualistic, because they considered matter and field as essentially different, the ideal would be to assume the particles to be point charges; this was impossible because of the infinite self-energy. Therefore it was necessary to assume the electron having a finite diameter and to make arbitrary assumptions about its inner structure, which lead to the difficulties pointed out in the introduction. In our theory these difficulties do not appear. We have seen that the p_{kl} field (or **D**-field) has a singularity which corresponds to a point charge as the source of the field. **D** and **E** are identical only at large distances ($r \gg r_0$) from the point charge, but differ in its neighbourhood, and one can call their quotient (which is function of **E**) "dielectric constant" of the space. But we shall now show that another interpretation is also possible which corresponds to the old idea of a spatial distribution of charge in the electron. It consists in taking div **E** (instead of div **D** $= 0$) as definition of charge density ρ, which we propose to call "free charge density."

Let us now write our set of field equations in the following form:

$$\frac{\partial \sqrt{-g}\, p^{kl}}{\partial x^l} = 0, \tag{3.4}$$

$$\frac{\partial f_{kl}}{\partial x^m} + \frac{\partial f_{lm}}{\partial x^k} + \frac{\partial f_{mk}}{\partial x^l} = 0, \quad \text{or} \quad \frac{\partial \sqrt{-g} f^{*kl}}{\partial x^l} = 0. \tag{3.2}$$

p^{kl} is a given function of f^{kl} and if we put in (3.4) for p^{kl} the expression (3.3), in which L is not specified, we obtain:

$$\frac{\partial}{\partial x^l}\left\{\left(2\frac{\partial \mathsf{L}}{\partial \mathrm{F}} f^{kl} + \frac{\partial \mathsf{L}}{\partial \mathrm{G}} f^{*kl}\right)\sqrt{-g}\right\} = 0. \tag{7.1}$$

We can now write the equation (7.1) in the form:

$$\frac{\partial \sqrt{-g} f^{kl}}{\partial x^l} = 4\pi \rho^k \sqrt{-g}, \tag{7.2}$$

where

$$-4\pi\rho^k = \frac{1}{2\,\partial\mathfrak{L}/\partial\mathfrak{F}}\left\{2f^{kl}\frac{\partial}{\partial x^l}\left(\frac{\partial\mathfrak{L}}{\partial\mathfrak{F}}\right)+f^{*kl}\frac{\partial}{\partial x^l}\left(\frac{\partial\mathfrak{L}}{\partial\mathfrak{G}}\right)\right\}. \tag{7.3}$$

The equations (7.2) and (3.2) are formally identical with the equations of the Lorentz theory. But the important difference consists in this, that ρ^k is *not* a given function of the space-time co-ordinates, but is a function of the unknown field strength. If we have a solution of our set of equations, we are able to find the density of the "free charge" or the "free current" with help of (7.2) or (7.3).

We see immediately that ρ^k satisfies the conservation law:

$$\frac{\partial\sqrt{-g}\,\rho^k}{\partial x^k} = 0. \tag{7.4}$$

This follows from (7.2), that is from the antisymmetrical character of f^{kl}, and can also be checked from (7.3).

In Lorentz's theory there exists the energy-impulse tensor of the electromagnetic field, defined by

$$4\pi S_i^{\ l} = \tfrac{1}{2}\delta_i^{\ l}\,\mathfrak{F} - f^{ls}f_{ks}, \tag{7.5}$$

but its divergence does not vanish, where the density of charge is not zero. Therefore to preserve the conservation principle in the Lorentz's theory it was necessary to introduce an energy-impulse tensor of matter, M^{kl}, the meaning of which is obscure. The tensor $M_i^{\ l}$ had to fulfil the condition that the divergence of $S_i^{\ l} + M_i^{\ l}$ vanishes. This difficulty does not appear in our theory. We do not need to introduce the matter tensor $M_i^{\ l}$ because the conservation laws are always satisfied by our energy-impulse tensor $T_k^{\ l}$.

We shall, however, show that it is possible by introducing the free charges to bring our conservation law

$$T^k_{\ s\,|\,k} = 0$$

into the form used in the Lorentz theory, namely

$$S^k_{\ s\,|\,k} = f^{sk}\rho_k. \tag{7.6}$$

The calculations are similar to those used in § 3. The simplest way is to choose a geodetic co-ordinate system. We have then:

$$\frac{\partial f^{kl}}{\partial x^l} = 4\pi\rho^k \tag{7.2A}$$

$$\frac{\partial f_{kl}}{\partial x^m}+\frac{\partial f_{lm}}{\partial x^k}+\frac{\partial f_{mk}}{\partial x^k} = 0. \tag{3.2}$$

Multiplying (3.2) by f^{kl} we find :

$$\tfrac{1}{2}\frac{\partial}{\partial x^s}(f_{kl}f^{kl}) - 2\frac{\partial}{\partial x^l}(f^{lk}f_{sk}) = 2f_{sk}\frac{\partial f^{lk}}{\partial x^l} \tag{7.7}$$

and therefore

$$\tfrac{1}{4}\frac{\partial}{\partial x^s}(f_{kl}f^{kl}) - \frac{\partial}{\partial x^l}(f^{lk}f_{sk}) = f_{sk}\rho^k, \tag{7.8}$$

and taking account of (7.5) :

$$\frac{\partial S_s^{\ k}}{\partial x^k} = f_{sk}\rho^k. \tag{7.9}$$

One can derive the same equation directly from the conservation formula (3.8) writing it, in a geodetic co-ordinate system, in the form

$$\frac{\partial T_s^{\ k}}{\partial x^k} = 0, \tag{3.8}$$

and introducing the expression (3.6) for $T_s^{\ k}$. The two methods are equivalent.

Let us now specialize our equations for the case in which L has the form given in (2.15). We obtain then for ρ^k in (7.3)

$$-4\pi\rho^k = \frac{1}{\sqrt{1+F-G^2}}\left\{f^{kl}\frac{\partial}{\partial x^l}\left(\frac{1}{\sqrt{1+F-G^2}}\right) - f^{*kl}\frac{\partial}{\partial x^l}\left(\frac{1}{\sqrt{1+F-G^2}}\right)\right\}. \tag{7.10}$$

In the space-vector notation, where

$$(\rho_1, \rho_2, \rho_3, \rho_4) \to \left(\frac{1}{c}, \rho\right),$$

we have

$$\left.\begin{aligned} -4\pi\frac{1}{c} &= \frac{1}{\sqrt{1+F-G^2}}\left\{\mathbf{B}\times\operatorname{grad}\left(\frac{1}{\sqrt{1+F-G^2}}\right)\right. \\ &\qquad \left. -\mathbf{E}\times\operatorname{grad}\left(\frac{G}{\sqrt{1+F-G^2}}\right)\right\} \\ &\quad + \frac{1}{c\sqrt{1+F-G^2}}\left\{\mathbf{E}\frac{\partial}{\partial t}\left(\frac{1}{\sqrt{1+F-G^2}}\right)\right. \\ &\qquad \left. -\mathbf{B}\frac{\partial}{\partial t}\left(\frac{G}{\sqrt{1+F-G^2}}\right)\right\} \\ 4\pi\rho &= -\frac{1}{\sqrt{1+F-G^2}}\left\{\mathbf{E}\cdot\operatorname{grad}\left(\frac{1}{\sqrt{1+F-G^2}}\right)\right. \\ &\qquad \left. -\mathbf{B}\cdot\operatorname{grad}\left(\frac{1}{\sqrt{1+F-G^2}}\right)\right\} \end{aligned}\right\}. \tag{7.10A}$$

We shall now apply the results here obtained to the case of the statical field. In this case **E** is always finite and has a non-vanishing divergence, which represents the free charge. We can, therefore, regard an electron either as a point charge, i.e., as a source of the **D** (p_{kl}) field, or as a continuous distribution of the space charge which is a source of the **E** (f_{kl}) field. It can easily be shown that the whole charge is in both cases the same (as is to be expected). Both

$$\int \operatorname{div} \mathbf{D}\, dv \quad \text{and} \quad \int \operatorname{div} \mathbf{E}\, dv$$

have the same value, *i.e.*, $4\pi e$. For the first integral it has been shown in § 6. For the second we have

$$E_r \sim \frac{e}{r^2} \quad \text{as} \quad r \to \infty ;$$

everywhere else E_r is finite. The discontinuity of E_x, E_y, E_z at the origin is also finite and gives no contribution to the integral. Therefore

$$\int \operatorname{div} \mathbf{E}\, dv = 4\pi \int \rho\, dv = 4\pi e.$$

Let us now calculate the distribution of the free charge in the statical case. We could calculate it from the equation (7.10A), but it is easier to do it from the equation

$$\operatorname{div} \mathbf{E} = \frac{1}{r^2}\frac{d}{dr}(r^2 E_r) = 4\pi\rho, \tag{7.11}$$

where

$$E_r = \frac{e}{r_0^2}\frac{1}{\sqrt{1 + (r/r_0)^4}}. \tag{6.15}$$

The result is

$$\rho = \frac{e}{2\pi r_0^3}\frac{r}{r_0}\left(1 + \frac{r^4}{r_0^4}\right)^{-3/2}. \tag{7.12}$$

For $r = r_0$, $\rho \propto r^{-7}$, therefore diminishing very rapidly as r increases. For $r \ll r_0$, $\rho \propto 1/r$, therefore $\rho \to \infty$, but $r^2\rho \to 0$ for $r \to 0$. It is easy to verify that the space integral of ρ is equal to e. For one has, putting $r/r_0 = \sqrt{\tan\phi}$

$$\int \rho\, dv = \frac{2e}{r_0^3}\int_0^\infty \frac{r^2\, dr}{\frac{r}{r_0}\left(1 + \frac{r^4}{r_0^4}\right)^{3/2}} = e\int_0^{\pi/2} \cos\phi\, d\phi = e.$$

Our theory combines the two possible aspects of the field; true point charges and free spatial densities are entirely equivalent. The question whether the one or the other picture of the electron is right has no meaning. This confirms the idea which has proved so fruitful in quantum mechanics, that one has to be careful in applying notions from the macroscopic world to the world of atoms: it may happen that two notions contradictory in macroscopic use are quite compatible in microphysics.

§ 8. *Lorentz's Equations of Point Motion and Mass.*

We consider once more the problem of the electron at rest. We intend to calculate the mass and to determine the absolute field constant b in terms of observable quantities. It is convenient to use the space vector notation.

The impulse-energy tensor is according to (3.6B)

$$\left.\begin{aligned} 4\pi X_x &= -D_x E_x - b^2 L = -\frac{E_x^2}{\sqrt{1-1/b^2\mathbf{E}^2}} \\ &\qquad - b^2\left(\sqrt{1-\frac{1}{b^2}\mathbf{E}^2}-1\right) \\ 4\pi X_y &= -D_x E_y = -\frac{E_x E_y}{\sqrt{1-1/b^2\mathbf{E}^2}} \\ S_x &= S_y = S_z = 0 \\ 4\pi U &= \mathbf{D}\,.\,\mathbf{E} + b^2 L = b^2 H = b^2\left(\sqrt{1-\frac{1}{b^2}\mathbf{E}^2}-1\right) \\ &\qquad + \frac{\mathbf{E}^2}{\sqrt{1-1/b^2\mathbf{E}^2}} = b^2\left(\frac{1}{\sqrt{1-1/b^2\mathbf{E}^2}}-1\right) \end{aligned}\right\} \quad (3.6c)$$

We calculate the space integrals of these quantities. Obviously one has with $dv = dx\,dy\,dz$:

$$4\pi\int X_x\,dv = 4\pi\int Y_y\,dv = 4\pi\int Z_z\,dv = -\int\frac{\mathbf{E}^2}{\sqrt{1-1/b^2\mathbf{E}^2}}\,dv - b^2\int\left(\sqrt{1-\frac{1}{b^2}\mathbf{E}^2}-1\right)dv \quad (8.1)$$

$$\int X_y\,dv = \int X_z\,dv = \int Z_y\,dv = 0. \quad (8.2)$$

Using (6.15) and (6.8) we find:

$$\int X_x\,dv = b^2(I_1 - I_2), \quad (8.3)$$

where

$$\left.\begin{aligned} I_1 &= \int_0^\infty \left(1 - \frac{x^2}{\sqrt{1+x^4}}\right) x^2\,dx \\ I_2 &= \tfrac{1}{3}\int_0^\infty \frac{dx}{\sqrt{1+x^4}} = \tfrac{1}{3} f(0) \end{aligned}\right\}. \tag{8.4}$$

The integral I_1 can be transformed by partial integration:

$$I_1 = \tfrac{1}{3}\int_0^\infty \left(1 - \frac{x^2}{\sqrt{1+x^4}}\right)\frac{dx^3}{dx}\,dx = \left[\tfrac{1}{3}\left(1 - \frac{x^2}{\sqrt{1+x^4}}\right)x^3\right]_0^\infty + \tfrac{2}{3}\int_0^\infty \frac{x^4 dx}{(1+x^4)^{3/2}}$$

The first term vanishes; the second can be transformed by another partial integration:

$$I_1 = -\tfrac{1}{3}\int_0^\infty x\,\frac{d}{dx}\left(\frac{1}{\sqrt{1+x^4}}\right)dx = \tfrac{1}{3}\int_0^\infty \frac{dx}{\sqrt{1+x^4}} = I_2 = \tfrac{1}{3} f(0).$$

The result is the so-called "theorem of Laue"†

$$\int X_x\,dv = \int Y_y\,dv = \int Z_z\,dv = 0.$$

In the statical case and in a co-ordinate system in which the electron is at rest the integrals of all components of the tensor T_k^l vanish except the total energy

$$E = \int U\,dv = \frac{b^2}{4\pi}\int H\,dv. \tag{8.5}$$

We find from (3.6c), (6.15), and (8.4)

$$E = m_0 c^2 = \delta^2 r_0^3 (3I_2 - I_1) = \frac{e^2}{r_0} 2I_1 = \frac{8}{3}\frac{e^2}{r_0} f(0) = 1{\cdot}2361\,\frac{e^2}{r_0}. \tag{8.6}$$

We have obtained a *finite value of the energy or the mass of the electron* with a definite numerical factor. This relation enables us to complete our theory concerning the value of the absolute field b in the conventional units. For (8.6) gives the "radius" of the electron expressed in terms of its charge and mass:

$$r_0 = 1{\cdot}2361\,\frac{e^2}{m_0 c^2} = 2.28 \times 10^{-13}\ \text{cm}. \tag{8.7}$$

and

$$b = \frac{e}{r_0^2} = 9.18 \cdot 10^{15}\ \text{e.s.u.} \tag{8.8}$$

The enormous magnitude of this field justifies the application of the Maxwell's

† Mie, 'Ann. Physik,' vol. 40, p. 1 (1913).

equations in their classical form in all cases, except those where the inner structure of the electron is concerned (field of the order b, distance or wavelength of the order r_0).

It can be shown that the motion of an elementary charge, on which an external field is acting, satisfies an equation which is an obvious generalization of the classical equation of Lorentz. To find this equation we shall use here a cartesian co-ordinate system.

We assume that the strength of the external field in a region surrounding the electron is very small compared with the proper field of the point charge. We denote the proper field of the electron by

$$p_{kl}^{(0)}, \quad f_{kl}^{(0)}, \tag{8.9}$$

and the external field by

$$p_{kl}^{(e)} = f_{kl}^{(e)}; \tag{8.10}$$

we do not take into consideration the sources of the external field. From the assumption, that

$$p_{kl}^{(0)} \gg f_{kl}^{(e)}; \quad f_{kl}^{(0)} \gg f_{kl}^{(e)} \tag{8.11}$$

inside the sphere surrounding the electron, it follows evidently that the real solution of the field equations cannot be very different from that obtained by adding the unperturbed proper field and the external field. We construct therefore a sphere $S^{(0)}$ with its centre at the singularity of H and with a radius r^0, which is so small, that inside the sphere (8.11) is always satisfied. But the radius r^0 of the sphere has to be great compared with the radius of the electron, so that we can assume the validity of Maxwell's equations on the surface of the sphere just as outside the sphere.

We make the further assumption that the acceleration (curvature of the world line) is not too large, *i.e.*, one can choose the radius in such a way that the field $p_{kl}^{(0)}$ inside $S^{(0)}$ is essentially identical with that of the charge e in uniform motion and can be derived from the formula of § 7 by a Lorentz transformation. Now we split the integral

$$\int \mathsf{H} d\tau \tag{8.12}$$

into a part corresponding to the sphere $S^{(0)}$ and the rest of space R. In $S^{(0)}$ we have

$$\left.\begin{aligned} \mathsf{H} &= \sqrt{1 - \tfrac{1}{2} p_{kl}\, p^{kl}} - 1 \\ &= \sqrt{1 - \tfrac{1}{2} p_{kl}^{(0)}\, p^{(0)\,kl} - p_{kl}^{(0)}\, f^{(e)kl} - \tfrac{1}{2} f_{kl}^{(e)}\, f^{(e)kl}} - 1 \\ \mathsf{Q} &= 0 \end{aligned}\right\} \tag{8.13}$$

Corresponding to (8.11) one can consider the terms $f_{kl}^{(0)} f^{(e)kl}$ as small of the first order (compared with $f_{kl}^{(0)} f^{(0)kl}$), the terms $f_{kl}^{(e)} f^{(e)kl}$ as small of the second order, and these latter will be neglected. Then we have by developing (8.13) and using (4.14):

$$\mathsf{H} = \sqrt{1 - \tfrac{1}{2} p_{kl}^{(0)} p^{(0)kl}} - 1 - \tfrac{1}{2} f_{kl}^{(0)} f^{(e)kl}, \tag{8.14}$$

which holds inside the sphere $S^{(0)}$. We can write (8.14) in another form:

$$\left.\begin{aligned} \mathsf{H} &= \mathsf{H}^{(0)} - \tfrac{1}{2} f_{kl}^{0} f^{(e)kl} - \tfrac{1}{4} f_{kl}^{(e)} f^{(e)kl} \\ \mathsf{H}^{(0)} &= \sqrt{1 - \tfrac{1}{2} p_{kl}^{(0)} p^{(0)kl}} - 1 \end{aligned}\right\} \tag{8.15}$$

(8.15) differs from (8.14) only in the terms of the second order. But (8.15) holds not only inside but also outside the sphere. For in R the equation (8.15) takes, according to our assumptions about $r^{(0)}$, the following form:

$$\begin{aligned} \mathsf{H} &= -\tfrac{1}{4} p_{kl}^{(0)} p^{(0)kl} - \tfrac{1}{2} f_{kl}^{(0)} f^{(e)kl} - \tfrac{1}{4} f_{kl}^{(e)} f^{(e)kl} \\ &= -\tfrac{1}{4} f_{kl}^{(0)} f^{(0)kl} - \tfrac{1}{2} f_{kl}^{(e)} f^{(0)kl} - \tfrac{1}{4} f_{kl}^{(e)} f^{(e)kl}. \end{aligned} \tag{8.16}$$

This is, however, the known expression for H in Maxwell's theory; $(\mathsf{L} = -\mathsf{H})$. Therefore (8.15) holds as well in the sphere $S^{(0)}$ as in R. One has

$$\int \mathsf{H}\, d\tau = \int \mathsf{H}^{(0)}\, d\tau - \tfrac{1}{2} \int f_{kl}^{(0)}\, f^{(e)kl}\, d\tau - \tfrac{1}{4} \int f_{kl}^{(e)} f^{(e)kl}\, d\tau. \tag{8.17}$$

We introduce the notation

$$4\pi\Lambda = \int \mathsf{H}^{(0)}\, dv - \tfrac{1}{2} \int f_{kl}^{(0)} f^{(e)kl}\, dv - \tfrac{1}{4} \int f_{kl}^{(e)} f^{(e)kl}\, dv, \tag{8.18}$$

and have for the action principle

$$\delta \int \Lambda dt = 0. \tag{8.19}$$

The integral

$$\int f_{kl}^{(e)} f^{(e)kl}\, d\tau \tag{8.20}$$

in (8.17) gives zero, because

$$\frac{\partial f^{(e)kl}}{\partial x^l} = 0\,; \quad (\rho_k^{(e)} = 0). \tag{8.21}$$

If we bear in mind that in the co-ordinate system, where the point charge is at rest, $\int \mathsf{H}^{(0)}\, dv$ is proportional to the mass, we have:

$$\int \mathsf{H}^{(0)}\, dv = m_0 c^2 \int \sqrt{1 - \mathfrak{v}^2/c^2}\, dv, \tag{8.22}$$

where $\mathbf{v}$ is the velocity of the centre of the electron. In the second integral of (8.17) we have

$$f_{kl}^{(e)} = \frac{\partial \phi_l^{(e)}}{\partial x^k} - \frac{\partial \phi_k^{(e)}}{\partial x^l} \tag{8.23}$$

and by partial integration we find, using $\frac{\partial f^{(0)kl}}{\partial x^l} = 4\pi\rho^k$:

$$\tfrac{1}{2}\int f^{0kl} f_{kl}^{(e)}\, dv\, dt = -4\pi \int \phi_l^{(e)} \rho^l \, dv\, dt. \tag{8.24}$$

The additional surface integral over the infinitely large surface can be omitted, because it gives no contribution to the variation (8.19). The result is:

$$\Lambda = m_0 c^2 \sqrt{1 - \mathbf{v}^2/c^2} - \int \phi_l^{(e)} \rho^l \, dv. \tag{8.25}$$

We can write (8.25) in the space-vector form:

$$\Lambda = m_0 c^2 \sqrt{1 - \mathbf{v}^2/c^2} - \int \phi^{(e)} \rho \, dv + \frac{1}{c}\int \mathbf{A}\mathbf{I}\, dv. \tag{8.25A}$$

An electron behaves therefore ~~like a mechanical system~~† with the rest mass m_0, acted on by the external field $f_{kl}^{(e)}$.‡

If the external potential is essentially constant in a region surrounding the electron considered, the diameter of which is large compared with r_0, one gets instead of (8.25):

$$\int \Lambda\, dt = \int m_0 c^2 \sqrt{1 - \mathbf{v}^2/c^2} + e\,(\phi^{(e)} - \mathbf{v}\,\mathbf{A}^{(e)}/c), \tag{8.26}$$

and this is entirely equivalent to Lorentz's equations of motion. But our

† Born, 'Ann. Physik,' vol. 28, p. 571 (1909); Pauli, "Relativitätstheorie," p. 642 (Teubner).

‡ The method used in I for deriving the equation of motion is not correct. It started from the action principle in the form

$$\delta \int \mathrm{L}\, d\tau = 0 \qquad (\text{instead } \delta \int \mathrm{H}\, d\tau = 0);$$

then in the development instead of the coefficients $f_{kl}^{(0)}$ the $p_{kl}^{(0)}$ appear, which become infinite at the centre of the electron. Therefore the transformation of the space integral is not allowed. In the first approximation we have

$$p_{kl} = p_{kl}^{(0)} + p_{kl}^{(e)}$$

and *not*

$$f_{kl} = f_{kl}^{(0)} + f_{kl}^{(e)}.$$

The mistake in the former derivation is also shown by the wrong result for the mass (the numerical factor was half of that given here).

formula (8.25) holds also for fields which are not constant. Any field can be split up in Fourier components or elementary waves; we may consider each of those separately, and choosing the Z-axis parallel to the propagation of the wave, we can assume that $\phi_s^{(e)}$ is proportional to $e^{2\pi i z/\lambda}$. Then we see that this Fourier component gives a contribution to the integral (8.25) of the form (8.26), where $\phi_s^{(e)}$ is now the amplitude of this component and e has to be replaced by an "effective" charge $\bar{e}$, given by

$$\bar{e} = \int \rho e^{2\pi i z/\lambda} d\tau.$$

Using the expression of ρ given by (7.12), and putting $z = r\cos\vartheta$,

$$d\tau = r^2 \sin\vartheta \, d\vartheta \, d\phi \, dr,$$

one has

$$\bar{e} = \frac{e}{r_0^3} \int_0^{\infty} \int_0^{\pi} \frac{r^2 \, dr}{r_0 \left(1 + \frac{r^4}{r_0^4}\right)^{3/2}} e^{\frac{2\pi i r}{\lambda} \cos\vartheta} \sin\vartheta \, d\vartheta .$$

The ϑ integration can be performed, and one can write

$$\bar{e} = e\, g\left(\frac{2\pi r_0}{\lambda}\right); \quad g(x) = \frac{2}{x} \int_0^x \frac{\sin xy}{(1+y^4)^{3/2}} \, dy.$$

For waves long compared with r_0 one has $\bar{e} = e$, because $g(0) = 1$. But for decreasing wave-lengths the effective charge diminishes, as the little table for $g(x)$ shows:

Table of $g(x)$.†

x.	$g(x)$.	x.	$g(x)$.
0	1	1·25	0·796
0·1	0·988	1·50	0·730
0·2	0·984	1·75	0·659
0·3	0·968	2·00	0·588
0·4	0·959	2·25	0·526
0·5	0·949	2·50	0·457
0·6	0·929	3·00	0·347
0·7	0·917	3·50	0·252
0·8	0·901	4·00	0·186
0·9	0·880	5·00	0·094
1·0	0·856		

The decrease begins to become remarkable where $x \sim 1$, or $\lambda \sim 2\pi r_0$. For large x one has $g(x) \cong 2/x^2$.

† Calculated by Mr. Devonshire.

If we introduce the quantum energy corresponding to the wave-length λ by $E = hc/\lambda$, then using (8.6) one has

$$x = \frac{2\pi r_0}{\lambda} = 1{\cdot}236\,\frac{2\pi}{hc}\,\frac{E}{m_0c^2} = \frac{1{\cdot}236}{137{\cdot}1}\,\frac{E}{m_0c^2} = \frac{1}{111}\,\frac{E}{m_0c^2}.$$

$x = 1$ corresponds to a quantum energy of about $100\,m_0c^2 = 5\,.\,10^7$ e. volt. For energies larger than this the interaction of electrons with other electrons (or light waves excited by those) should become smaller than that calculated by the accepted theories. This consequence seems to be confirmed by the astonishingly high penetrating power of the cosmic rays.†

Summary.

The new field theory can be considered as a revival of the old idea of the electromagnetic origin of mass. The field equations can be derived from the postulate that there exists an "absolute field" b which is the natural unit for all field components and the upper limit of a purely electric field. From the standpoint of relativity transformations the theory can be founded on the assumption that the field is represented by a non-symmetrical tensor a_{kl}, and that the Lagrangian is the square root of its determinant; the symmetrical part g_{kl} of a_{kl} represents the metric field, the antisymmetrical part f_{kl} the electromagnetic field. The field equations have the form of Maxwell's equations for a polarizable medium for which the dielectric constant and the magnetic susceptibility are special functions of the field components. The conservation laws of energy and momentum can be derived. The static solution with spherical symmetry corresponds to an electron with finite energy (or mass); the true charge can be considered as concentrated in a point, but it is also possible to introduce a free charge with a spatial distribution law. The motion of the electron in an external field obeys a law of the Lorentz type where the force is the integral of the product of the field and the free charge density. From this follows a decrease of the force for alternating fields of short wave-lengths (of the order of the electronic radius), in agreement with the observations of the penetrating power of high frequency (cosmic) rays.

† Born, 'Nature,' vol. 133, p. 63 (1934).