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Influence of Dissipation on Quantum Tunneling in Macroscopic Systems

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A quantum system which can tunnel, at $T = 0$, out of a metastable state and whose interaction with its environment is adequately described in the classically accessible region by a phenomenological friction coefficient η, is considered. By only assuming that the environment response is linear, it is found that dissipation multiplies the tunneling probability by the factor $\exp[-A\eta(\Delta q)^2/\hbar]$, where Δq is the "distance under the barrier" and A is a numerical factor which is generally of order unity.

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One of the more intriguing prospects opened up by recent advances in cryogenics is the possibility of observing quantum tunneling on a macroscopic scale.[1] Generally, we expect tunneling to be the predominant decay mode of a metastable state when $k_B T \ll \hbar\omega_0$, where ω_0 is the frequency of small oscillations about the metastable equilibrium; and there exist macroscopic systems for which this condition can be satisfied while the tunneling probability is not unobservably small. One particularly promising candidate[2] is a SQUID[3] (superconducting quantum interference device); in this case the relevant macroscopic variable is the magnetic flux trapped in the ring, and a straightforward WKB calculation ignoring dissipation[4] predicts that for typical SQUID parameters quantum tunneling should become the dominant flux transition mechanism for $T \lesssim 100$ mK. Indeed, two recent experiments[5,6] at even higher temperatures (~ 1–2 K) have been interpreted as possible evidence for quantum tunneling of the flux. Whether or not this interpretation is correct, the observation of such a phenomenon would clearly be of intrinsic interest for the extrapolation of quantum mechanics to the macroscopic scale.[5]

An important qualitative difference between quantum tunneling in macroscopic systems and its experimentally well-verified microscopic analog lies in the relative importance and nature of the coupling to the environment. For microscopic systems this coupling is often negligible and, even when it is not, can usually be described by a well-defined Hamiltonian and treated in low-order perturbation theory (as, for example, in the theory of inelastic electron tunneling in metal-insulator junctions[7]). On the other hand, in macroscopic systems the coupling is often so strong that the motion in the classically accessible region is highly damped or even (as in most practical SQUID's) overdamped; moreover, we are often ignorant of the precise details of the coupling and are reduced to describing its effects by phenomenological friction or viscosity coefficients whose values must be taken from experiment. The object of this Letter is to develop a theory of the quantum tunneling process which will take these factors into account. There is space here only to give the general outlines of this theory; we intend to give a more extended discussion elsewhere, including the specific application to SQUID's. (See also Caldeira, Ref. 4.)

We confine ourselves to the case of zero temperature and consider a system characterized by a macroscopic coordinate q with which is associated a smooth potential-energy function $V(q)$ with a metastable minimum; we choose the axes so that this lies at the origin ($q=0$, $V=0$). In what follows we denote the height of the barrier separating the metastable potential minimum from regions of lower potential by V_0 and its "width" [that is, the first nonzero value of q for which $V(q)=0$] by Δq. The "mass" of the system is denoted by M and the frequency of small oscillations around the metastable equilibrium, $[M^{-1} V''(0)]^{1/2}$, by ω_0; we assume that $\omega_0 \ll V_0/\hbar$ and will work only to lowest order in the quantity[8] $\exp(-V_0/\hbar\omega_0)$. The system is assumed to be coupled to its environment in a way which is not necessarily known in detail, but which has the consequence that when the energy E satisfies the condition $V_0 - E \gg \hbar\omega_0$ the expectation value of $q(t)$ obeys, at least approximately, a classical equation of motion with friction coefficient η, i.e.,

$$M\ddot{q} + \eta\dot{q} = -dV/dq + F_{\text{ext}}(t). \qquad (1)$$

In particular, for $E \ll V_0$ and $F_{\text{ext}}=0$ the system undergoes simple damped harmonic motion with damping $\gamma \equiv \eta/2M$. We shall assume in what follows that the coupling to the environment is "weak" in the sense that the response of the latter to the system may be treated as linear; however, it is essential to appreciate that this does *not* imply the condition $\gamma \ll \omega_0$; in fact, it is quite compatible with the opposite limit of strong overdamping ($\gamma \gg \omega_0$). We then pose the question: How does the "friction" affect the probability of quantum tunneling out of the metastable ground state?

Our principal result is this: The effect of (linear) friction is to multiply the tunneling probability calculated in its absence by a factor $\exp[-A\eta \times (\Delta q)^2/\hbar]$, where Δq is, as above, the "distance under the barrier" and A is a numerical factor which depends weakly on the ratio γ/ω_0 and is of order 1 to within logarithmic factors.

Our method is an extension of one originally used by Langer[9] in the context of the thermal nucleation problem and further developed by Stone[10] and by Callan and Coleman[11]; we refer to these papers for details of the relevant analytic continuation procedures, etc., which are precisely parallel in the damped and undamped cases. Denoting the set of environment coordinates by $\{x_\alpha\}$, we define the reduced "imaginary-time" Green's function of the system by the expression

$$\tilde{K}(q_i, q_f; \tau) \equiv \int \prod_\alpha dx_{\alpha i} K(q_i, q_f; \{x_{\alpha i}\}, \{x_{\alpha f}\}; \tau)_{x_{\alpha i} = x_{\alpha f}}, \qquad (2)$$

where $K(q_i, q_f; \{x_{\alpha i}\}, \{x_{\alpha f}\}; \tau)$ is the quantum-mechanical transition amplitude for the "universe" (system plus environment) to go from coordinates $(q_i, \{x_{\alpha i}\})$ at time zero to $(q_f, \{x_{\alpha f}\})$ at time τ, and is given by the Feynman path integral

$$K(q_i, q_f; \{x_{\alpha i}\}, \{x_{\alpha f}\}; \tau) = \int_{q(0)=q_i}^{q(t)=q_f} \mathfrak{D}q(t) \int_{\{x_\alpha(0)\}=\{x_{\alpha i}\}}^{\{x_\alpha(\tau)\}=\{x_{\alpha f}\}} \prod_\alpha \mathfrak{D}x_\alpha(t) \exp\Big(-\int_0^\tau L_{\text{E}}(q(t), \{x_\alpha(\tau)\}) dt/\hbar\Big), \qquad (3)$$

where L_{E} is the "Euclidean" Lagrangian, i.e. (kinetic energy) $+V(q, \{x_\alpha\})$. The quantity $\tilde{K}(q_i, q_f; \tau)$ has the spectral expansion

$$\tilde{K}(q_i, q_f; \tau) = \sum_n \int \prod_\alpha dx_\alpha \psi_n^*(q_i, \{x_\alpha\}) \psi_n(q_f, \{x_\alpha\}) \exp(-E_n \tau/\hbar) \qquad (4)$$

and an inspection of the quantity $\tilde{K}(q,q;\tau)$ for small q in the limit $\tau \to \infty$ therefore gives both the probability density and the energy of the metastable ground state. In particular, the resulting E_n will have (after the appropriate analytic continuation procedures) a very small imaginary part which gives us the quantum tunneling rate.

To obtain a useful expression for $\tilde{K}(q_i, q_f; \tau)$ we must exploit our assumption that the response of the environment is linear [at least for the amplitudes of $q(t)$ important in the quantum tunneling process]. Since any system whose response is linear can be represented by a set of harmonic oscillators (and since by hypothesis the friction is linear in q), we may without loss of generality write the Euclidean Langrangian for the coupled system and environment in the form[12]

$$L_{\text{E}} = \tfrac{1}{2}M\dot{q}^2 + V(q) + \tfrac{1}{2}\sum_\alpha m_\alpha \dot{x}_\alpha^{\,2} + \tfrac{1}{2}\sum_\alpha m_\alpha \omega_\alpha^{\,2} x_\alpha^{\,2} + q\sum_\alpha c_\alpha x_\alpha, \qquad (5)$$

where m_α, ω_α, and c_α are parameters which we do not need to know in detail (see below). The functional integrals over $x_\alpha(t)$ in Eq. (3) and the integrals over $x_{\alpha i}$ in Eq. (2) can now be (somewhat tediously) performed[13] and yield, in the limit $\tau \to \infty$,

$$\tilde{K}(q_i, q_f; \tau) = \int_{q(0)=q_i}^{q(\tau)=q_f} \mathfrak{D} q(t) \exp[-S_{\rm eff}\{q(t)\}/\hbar], \tag{6}$$

where the "effective action" $S_{\rm eff}\{q(t)\}$ is given by

$$S_{\rm eff}\{q(t)\} = \int_0^\tau [\tfrac{1}{2}M\dot{q}^2 + V(q)]dt - \int_{-\infty}^{\infty}\int_0^\tau dt\,dt'\,\alpha(t-t')q(t)q(t') + \text{const}, \tag{7}$$

$$\alpha(t-t') \equiv \sum_\alpha (c_2{}^2/4m_\alpha\omega_\alpha)\exp(-\omega_\alpha|t-t'|) \equiv (1/2\pi)\int_0^\infty J(\omega)\exp(-\omega|t-t'|)d\omega \geq 0. \tag{8}$$

In Eq. (7) $q(t)$ is to be defined[13] outside the region $0<t<\tau$ by the prescription $q(t+\tau) \equiv q(t)$. (This is irrelevant to the calculation of the semiclassical tunneling exponent but does affect other quantities.)

It is convenient to rewrite the second term in Eq. (7) with use of the identity $q(t)q(t') \equiv \frac{1}{2}[q^2(t) + q^2(t')] - \frac{1}{2}[q(t) - q(t')]^2$. Then the squared terms can be lumped into $V(q)$ and have the effect of shifting the small oscillation frequency ω_0 downwards.[12] Since this shift occurs in the classically allowed motion as well as in the quantum tunneling, it is unobservable and we shall simply imagine that it has been incorporated in the definition of $V(q)$. Absorbing also the constant in (8) (the zero-point energy of the environment) into the zero of $V(q)$, we see that the remaining correction to $S_{\rm eff}$ is always positive.

To proceed further we need to relate the quantity $\alpha(t-t')$ defined in Eq. (8) to the phenomenological viscosity η. We first note that since the characteristic times in the "bounce trajectory" (see below) are in general of order $\omega_0{}^{-1}$, or longer, we need $\alpha(t-t')$ only for times of this order, or equivalently $J(\omega)$ for frequencies $\lesssim \omega_0$. Now, if the classical motion is to be determined by a well-defined friction coefficient at all (i.e., if the frictional force is to be proportional to the velocity), we must have in this frequency region the simple relation

$$J(\omega \leq \omega_c) = \eta\omega. \tag{9}$$

In the weak-damping limit this relation may be obtained simply by considering a large-amplitude classical motion of the system and equating the phenomenological expression for the power dissipated by it into the environment, $\eta\dot{q}^2$, to the quantum-mechanical golden-rule expression written in terms of J; in the more general case, it follows from a comparison of the ground-state probability distribution in the "harmonic" region $V(q) \ll V_0$ as calculated from Eq. (6) with the known expression[14] for this quantity for a linear damped harmonic oscillator with friction coefficient η. In view of Eqs. (8) and (9), we may, after the appropriate redefinition of the harmonic part of $V(q)$ (see above), write the expression for $S_{\rm eff}$ in the following form which, if ω_c is the frequency at which $J(\omega)$ deviates appreciably from its low-frequency form, is valid to lowest order in ω_0/ω_c:

$$S_{\rm eff}\{q(t)\} = \int_0^\tau [\tfrac{1}{2}M\dot{q}^2 + V(q)]dt + (\eta/4\pi)\int_{-\infty}^{\infty}\int_0^\tau dt\,dt'\{[q(t) - q(t')]/(t-t')\}^2. \tag{10}$$

Equation (10), when inserted into Eq. (6), is the principal result of this Letter. From this point on the argument closely parallels that which is well known in the undamped case[10,11]; we follow here the notation of Ref. 11. The tunneling probability is associated with the saddlepoint of the action functional in function space which represents a possible classical trajectory ("bounce") in the *inverted* potential $\tilde{V}(q) \equiv -V(q)$, in which the system falls off its unstable potential maximum at $q=0$, moves off to a finite value, say q_b, of q, turns around and returns to the origin as[15] $\tau \to \infty$. That such a classical motion still exists and is still a saddlepoint in the function space, even in the presence of the last term in Eq. (9), follows from the simple observation that for small q_b the action is positive and proportional to $q_b{}^2$, whereas for sufficiently large q_b it is possible to obtain an arbitrarily large negative action by staying long enough in the region $V(q)>0$ (i.e., $q_b > \Delta q$). In complete analogy to the results of Refs. 10 and 11, the tunneling probability is given by

$$P_{\rm QM} = (B/2\pi\hbar)^{1/2}\Delta^{-1/2}\exp(-B/\hbar), \tag{11}$$

where Δ is the ratio of two determinants and B is the quantity $S_{\rm eff}\{q(t)\}$, Eq. (10), evaluated along the classical "bounce" trajectory. The quantity $\Delta^{-1/2}$ has the dimensions of frequency; it evidently depends on η, but does not contain $\hbar$ and within the accuracy of the present calculation may be set of the order of the undamped frequency ω_0. The effects considered here are entirely associated with the dependence of the exponent B on η, which is likely to be the overwhelmingly dominant effect in cases of practical interest.

In the limit of weak damping the correction ΔB to B is obtained simply by evaluating the last term in Eq. (9) along the undamped trajectory $q(t) = (\Delta q)f(t)$, where $f(t)$ is zero at $\pm\infty$ and reaches a maximum value of 1. This gives

$$\Delta B = A_0 \eta (\Delta Q)^2, \quad A_0 \equiv (1/4\pi)\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt' [f(t) - f(t')]^2/(t-t')^2. \tag{12}$$

For the practically important case of a potential of the form $\frac{1}{2}\omega_0^2 q^2 - \frac{1}{3}\lambda q^2$ we have $A_0 = (12/\pi^3)\zeta(3)$.

In the general case it is easy to see that ΔB can be written in the form $\eta(\Delta q)^2\varphi(\alpha)$, where $\alpha \equiv \gamma/\omega_0$. There is space here only to quote without proof the result that, at any rate for any potential which is bounded by expressions of the form $aq^2 - bq^n$, $\varphi(\alpha)$ is bounded above by a constant and below by a function of α which is in general of order unity and tends to zero for large α as $(\ln\alpha)^{-1}$. We strongly suspect that $\varphi(\alpha)$ actually tends to zero for large α, if at all, even more slowly than logarithmically. At any rate the qualitative conclusion is clear: Linear friction suppresses quantum tunneling by a factor $\exp[-A\eta(\Delta q)^2/\hbar]$, where A is in general of order unity.

To take our results over to the case of a SQUID described by the "resistively shunted junction" model, it is only necessary to replace q in the above discussion by the trapped flux and η by the experimentally measurable[5] normal conductance of the junction (but cf. Ref. 12). We intend to discuss the details of this application elsewhere.

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