An extensible model of the electron

By P. A. M. Dirac, F.R.S.
St John's College, University of Cambridge

(*Received 5 February* 1962)

It is proposed that the electron should be considered classically as a charged conducting surface, with a surface tension to prevent it from flying apart under the repulsive forces of the charge. Such an electron has a state of stable equilibrium with spherical symmetry, and if disturbed its shape and size oscillate. The equations of motion are deduced from an action principle and a Hamiltonian formalism is obtained. The energy of the first excited state with spherical symmetry is worked out according to the Bohr–Sommerfeld method of quantization, and is found to be about 53 times the rest-energy of the electron. It is suggested that this first excited state may be considered as a muon. The present theory has no electron spin, so it cannot agree accurately with experiment.

The model

The concept of an electron of finite size is an old one, first proposed by Abraham and Lorentz. It is the most natural concept that makes the total energy of the Coulomb field of the electron finite.

Recently, new evidence has appeared for the finite size of the electron by the discovery of the muon, having properties so similar to the electron that it may be considered to be merely an electron in an excited state. If one works with a point-charge model of the electron, there is no place in the theory for the muon. However, if one supposes the electron to have a finite size, with no constraints fixing the size and shape, one can arrange that the variations of size and shape are stable oscillations about an equilibrium position, and then one can assume that the lowest excited state is the muon.

The present paper is concerned with developing the simplest model of this kind. The electron is assumed to have a charged conducting surface. Outside the surface Maxwell's equations hold. Inside the surface there is no field. To prevent the electron from flying apart under the Coulomb repulsion of its surface charge, a non-Maxwellian force is assumed of the type of a surface tension. So the electron may be pictured as a bubble in the electromagnetic field.

The classical equations of motion for this model will be worked out. They are applicable also to several electrons in interaction, provided no two of them ever come into contact.

Lees (1939) has set up a somewhat similar theory. Lees takes into account also the gravitational field, and instead of a surface tension he imposes constraints on the shape and size of the electron, which rule out the possibility of excited states.

A convenient way to secure the correct boundary conditions at the surface of the electron is to take the potentials to be zero inside and continuous at the surface. With several interacting electrons one can take the potentials to be zero inside all of them if one gives up the condition $\partial A^\mu/\partial x^\mu = 0$. The boundary conditions for the

field quantities just outside the surface are then the usual ones with a conducting surface, namely, the tangential components of the electric field and the normal component of the magnetic field are zero in a frame of reference for which the element of surface is instantaneously at rest.

The action principle

The equations of motion should be derivable from an action principle in order that they may be suitable for quantization. There should be just one comprehensive action principle, giving both the field equations outside the electron and equations of motion for the surface of the electron. The latter will provide equations of motion for the electron as a whole, as well as equations for the changes in the shape and size of the electron.

We shall work with a four-dimensional relativistic picture in which the surface of the electron appears as a tube with a three-dimensional surface. The action I consists of two terms, a four-dimensional integral I_0 extended over the space outside the tube and a three-dimensional integral I_s extended over the surface of the tube. The space inside the tube contributes no action.

We take I_0 to be the usual action for the Maxwell field,

$$4\pi I_0 = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}^4x,$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu},$$

where $\xi_{,\mu}$ denotes $\partial\xi/\partial x^\mu$. We want I_s to lead to forces like a surface tension, and we get this by taking it to be a constant times the three-dimensional 'area' of the tube.

To have an action principle, we must express the action I in terms of variables q which describe the physical conditions for all space and time and which are such that, when small variations are made in all the q's, δI is a linear function of the δq's.

The most obvious way of choosing the q's is as follows. To describe the surface of the tube, introduce parameters u_1, u_2, u_3 to specify a general point on it and then take as q's the four co-ordinates $x^\mu(u)$ of each point on it. Take also the four potentials A_μ throughout space-time as q's. There are then sufficient q's to fix the physical conditions completely, but δI is not a linear function of these δq's. If one makes a variation $\delta x^\mu(u)$ corresponding to the surface being pushed out a little, δI will not be minus the δI for $-\delta x^\mu(u)$, corresponding to the surface being pushed in a little, on account of the field just outside the surface being different from the field just inside. Thus this choice of q's will not do.

One can avoid the difficulty by working with general curvilinear co-ordinates x^μ. Let the surface of the tube in terms of these co-ordinates be $f(x) = 0$, with $f(x) > 0$ for the region outside. We now do not change $f(x)$ in the variation process, but produce an arbitrary variation of the surface by varying the co-ordinate system.

To get a convenient way of describing the co-ordinate system and its variation, introduce a second co-ordinate system y^A which is kept fixed during the variation

process and use the functions $y^{\Lambda}(x)$ to describe the x co-ordinate system in terms of the y co-ordinate system. To keep the two systems distinct, capital suffixes will be used to refer to the y system and small suffixes to refer to the x system. For simplicity, we take the y system to be rectilinear and orthogonal, so that the metric for the x system is

$$g_{\mu\nu} = y_{\Lambda,\mu} y^{\Lambda}{}_{,\nu}. \tag{1}$$

We now take as q's the $A_{\mu}(x)$ and $y^{\Lambda}(x)$ for all x's for which $f(x) \geqslant 0$. These q's, together with the fixed function $f(x)$, are sufficient to determine the equation of the surface relative to the y system and the field outside the surface relative to the y system. They thus determine everything of physical importance. They also determine something not of physical importance, namely, a curvilinear system of co-ordinates outside the surface. This is a complication, but it does not do any harm to the action principle. It appears to be unavoidable if one is to have δI linear in the δq's.

We may in principle choose the function f to be anything that can be fitted in with a continuous curvilinear system of co-ordinates x^{μ} for the space outside. If there is only one electron, the most convenient choice is

$$f(x) = x^1, \tag{2}$$

which can be fitted in with a deformed system of polar co-ordinates.

The boundary conditions are

$$A_{\mu}(x) = 0 \quad \text{for} \quad f(x) = 0. \tag{3}$$

They do not involve any derivatives, so they cannot appear as equations of motion following from the action principle. They must therefore be counted as constraints, whose validity is preserved during the variation process. With the choice (2) for f, they lead to the field conditions just outside the surface

$$F_{ab} = 0, \tag{4}$$

where a, b take on the values 0, 2, 3, here and in the future.

Expressed in terms of the new q's with the choice (2) for f, the action is

$$4\pi I_0 = -\frac{1}{4}\int_{x^1>0} J g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \, \mathrm{d}^4x, \tag{5}$$

$$4\pi I_s = -\omega \int_{x^1=0} M \, \mathrm{d}x^0 \, \mathrm{d}x^2 \, \mathrm{d}x^3, \tag{6}$$

where $-J^2$ is the determinant of the $g_{\mu\nu}$ and M^2 is the determinant of the g_{ab}, so that

$$M = J(-g^{11})^{\frac{1}{2}}. \tag{7}$$

ω is a positive constant that determines the equilibrium size and mass of the electron.

The equations of motion

The variation of $A_\mu(x)$ and $y^\Lambda(x)$ leads to the following, where four-dimensional integrals are over the region $x^1 > 0$ and three-dimensional integrals are over the surface $x^1 = 0$.

$$4\pi\delta I_0 = -\frac{1}{2}\int J\{g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}\delta F_{\rho\sigma} + g^{\mu\rho}F_{\mu\nu}F_{\rho\sigma}\delta g^{\nu\sigma} + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta}\delta g_{\alpha\beta}\}\,\mathrm{d}^4x$$

$$= \int J\{F^{\rho\sigma}\delta A_{\rho,\sigma} + \tfrac{1}{2}(F_{\mu\nu}F^{\mu}{}_{\sigma}g^{\nu\alpha}g^{\sigma\beta} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta})\,\delta g_{\alpha\beta}\}\,\mathrm{d}^4x$$

$$= \int J\{F^{\rho\sigma}\delta A_{\rho,\sigma} + (F_\mu{}^\alpha F^{\mu\beta} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta})\,y_{\Lambda,\alpha}\,\delta y^\Lambda{}_{,\beta}\}\,\mathrm{d}^4x$$

$$= -\int\{(JF^{\rho\sigma})_{,\sigma}\delta A_\rho + [J(F_\mu{}^\alpha F^{\mu\beta} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta})\,y_{\Lambda,\alpha}]_{,\beta}\delta y^\Lambda\}\,\mathrm{d}^4x$$

$$+\int J(F_\mu{}^\alpha F^{\mu 1} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha 1})\,y_{\Lambda,\alpha}\,\delta y^\Lambda\,\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^3 \qquad (8)$$

with the help of the boundary condition $\delta A_\rho = 0$. Let c^{ab} be the reciprocal matrix to g_{ab}. Then

$$4\pi\delta I_s = -\tfrac{1}{2}\omega\int Mc^{ab}\delta g_{ab}\,\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^3$$

$$= \omega\int (Mc^{ab}y_{\Lambda,a})_{,b}\,\delta y^\Lambda\,\mathrm{d}x^0\,\mathrm{d}x^2\,\mathrm{d}x^3. \qquad (9)$$

Equating to zero the coefficient of δA_ρ for $x^1 > 0$ in (8), we get

$$(JF^{\rho\sigma})_{,\sigma} = 0. \qquad (10)$$

These are the Maxwell equations for the outside space. Equating to zero the coefficient of δy^Λ for $x^1 > 0$ in (8), we get

$$[J(F_\mu{}^\alpha F^{\mu\beta} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta})\,y_{\Lambda,\alpha}]_{,\beta} = 0.$$

Multiplying by $y^\Lambda{}_{,\rho}$ and using (1), we can reduce it to

$$[J(F_\mu{}^\alpha F^{\mu\beta} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta})\,g_{\alpha\rho}]_{,\beta} - \tfrac{1}{2}J(F_\mu{}^\alpha F^{\mu\beta} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha\beta})\,g_{\alpha\beta,\rho} = 0,$$

or
$$(JF_{\mu\rho}F^{\mu\beta})_{,\beta} - \tfrac{1}{4}J(F_{\mu\nu}F^{\mu\nu})_{,\rho} - \tfrac{1}{2}JF_\mu{}^\alpha F^{\mu\beta}g_{\alpha\beta,\rho} = 0.$$

With the help of
$$(F_{\mu\nu}F^{\mu\nu})_{,\rho} = 2F^{\mu\beta}F_{\mu\beta,\rho} - 2F_\mu{}^\alpha F^{\mu\beta}g_{\alpha\beta,\rho}$$

it becomes
$$F_{\mu\rho}(JF^{\mu\beta})_{,\beta} + \tfrac{1}{2}JF^{\mu\beta}(F_{\mu\rho,\beta} - F_{\beta\rho,\mu} - F_{\mu\beta,\rho}).$$

It is thus a consequence of the Maxwell equations and does not give us anything new.

Finally, equating to zero the coefficient of δy^Λ on the surface in (8) plus (9), we get

$$J(F_\mu{}^\alpha F^{\mu 1} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^{\alpha 1})\,y_{\Lambda,\alpha} + \omega(Mc^{ab}y_{\Lambda,a})_{,b} = 0.$$

Multiplying this by $y^\Lambda{}_{,\rho}$, we get

$$J(F_{\mu\rho}F^{\mu 1} - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}g^1_\rho) + \omega(Mc^{ab}g_{a\rho})_{,b} = \omega Mc^{ab}y_{\Lambda,a}y^\Lambda{}_{,\rho b}$$
$$= \tfrac{1}{2}\omega Mc^{ab}g_{ab,\rho}$$
$$= \omega M_{,\rho}.$$

For $\rho \neq 1$, it reduces to $0 = 0$, on account of the surface conditions (4). For $\rho = 1$ it gives, with the help of

$$c^{ab} = g^{ab} - g^{1a}g^{1b}/g^{11}, \tag{11}$$

$$\begin{aligned} \tfrac{1}{2}F_{a1}F^{a1} &= \omega J^{-1}\{(Mg^{1b}/g^{11})_{,b} + M_{,1}\} \\ &= \omega J^{-1}(Mg^{1\mu}/g^{11})_{,\mu}. \end{aligned} \tag{12}$$

This is the equation of motion for the surface of the electron. The left-hand side may be written alternatively as $\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, on account of the boundary conditions (4). The right-hand side has a simple geometrical meaning. Apart from the factor ω it is just the total curvature of the surface.

The spherically symmetric solution

Let us apply the equation of motion (12) to a spherically symmetric electron whose centre is at rest, with no incident radiation. Let ρ be the radius of the electron, a function of the time t. We use polar co-ordinates and take $x^1 = r - \rho$, $x^2 = \theta$, $x^3 = \phi$, so that the equation of the surface is $x^1 = 0$. Then

$$\mathrm{d}s^2 = \mathrm{d}t^2 - (dx^1 + \dot{\rho}\,\mathrm{d}t)^2 - (x^1+\rho)^2\,\mathrm{d}\theta^2 - (x^1+\rho)^2\sin^2\theta\,\mathrm{d}\phi^2, \tag{13}$$

giving

$$g_{00} = 1-\dot{\rho}^2, \quad g_{11} = -1, \quad g_{10} = -\dot{\rho}$$

$$g_{22} = -(x^1+\rho)^2, \quad g_{33} = -(x^1+\rho)^2\sin^2\theta,$$

with the other components of $g_{\mu\nu}$ vanishing. This leads to

$$\begin{aligned} J &= (x^1+\rho)^2\sin\theta, \\ M &= (1-\dot{\rho}^2)^{\frac{1}{2}}(x^1+\rho)^2\sin\theta, \\ g^{00} &= 1, \quad g^{11} = -(1-\dot{\rho}^2), \quad g^{10} = -\dot{\rho}, \\ g^{22} &= -(x^1+\rho)^{-2}, \quad g^{33} = -(x^1+\rho)^{-2}\sin^{-2}\theta. \end{aligned}$$

We now find

$$\begin{aligned} \frac{1}{J}\left(\frac{Mg^{1\mu}}{g^{11}}\right)_{,\mu} &= \frac{1}{(x^1+\rho)^2}\frac{\partial}{\partial t}\frac{(x^1+\rho)^2\dot{\rho}}{(1-\dot{\rho}^2)^{\frac{1}{2}}} + \frac{(1-\dot{\rho}^2)^{\frac{1}{2}}}{(x^1+\rho)^2}\frac{\partial}{\partial x^1}(x^1+\rho)^2 \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{\rho}}{(1-\dot{\rho}^2)^{\frac{1}{2}}} + \frac{2}{\rho(1-\dot{\rho}^2)^{\frac{1}{2}}} \end{aligned}$$

on the surface $x^1 = 0$. Outside the electron the field is merely the Coulomb field, so just outside the surface $F_{\mu\nu}F^{\mu\nu} = e^2/\rho^4$. The equation of motion (12) thus becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{\rho}}{(1-\dot{\rho}^2)^{\frac{1}{2}}} = \frac{e^2}{2\omega\rho^4} - \frac{2}{\rho(1-\dot{\rho}^2)^{\frac{1}{2}}}. \tag{14}$$

Putting $\dot{\rho} = 0$, one gets for the equilibrium radius a of the electron

$$a^3 = e^2/4\omega. \tag{15}$$

For an electron instantaneously at rest the total energy E consists of two parts, the electrostatic energy of the Coulomb field, namely $e^2/2\rho$, and a surface tension energy proportional to ρ^2, say $\beta\rho^2$. Thus

$$E = e^2/2\rho + \beta\rho^2. \tag{16}$$

The minimum value of E must occur when $\rho = a$, so

$$e^2/2a^2 = 2\beta a,$$

giving $\beta = \omega$. We now see that in the equilibrium state the surface energy is half the electrostatic energy. Thus the total energy is $3e^2/4a$, which must equal m (with $c = 1$), so

$$a = 3e^2/4m. \tag{17}$$

For small oscillations about the equilibrium state equations (14) and (15) give

$$\ddot{\rho} = \frac{2a^3}{\rho^4} - \frac{2}{\rho} = \frac{2}{a}\left\{\left(1+\frac{\rho-a}{a}\right)^{-4} - \left(1+\frac{\rho-a}{a}\right)^{-1}\right\}$$
$$= -6(\rho-a)/a^2. \tag{18}$$

The frequency ν of the oscillations is thus

$$2\pi\nu = \sqrt{6}/a.$$

The energy of one quantum with this frequency is

$$h\nu = \sqrt{6}\hbar/a = (4\sqrt{6}/3)\, m\hbar/e^2 = 448m.$$

This is of the right order for the energy of the muon. However, the one-quantum oscillation is by no means a small one and the approximations we used in deriving (18) are not valid for it. To get a better theory we need the Hamiltonian formalism.

The general Hamiltonian

Going back to the general motion, we shall pass to the Hamiltonian form by the standard method. To simplify the work as much as possible, we shall consider only co-ordinate systems x^μ for which $x^0 = y^0$. This is permissible because it can be combined with (2) without imposing any restriction on the surface of the electron.

The formula (1) now reduces to

$$g_{\mu\nu} = g^0_\mu g^0_\nu + y_{R,\mu} y^R{}_{,\nu}. \tag{19}$$

(The suffixes, R, r, s take on the values 1, 2, 3.) From

$$g^{\mu\nu} = (\partial x^\mu/\partial y_\Lambda)\,(\partial x^\nu/\partial y^\Lambda)$$

we get $g^{00} = 1$, which shows that $J = K$, where $-K^2$ is the determinant of the g_{rs}.

The dynamical co-ordinates are $A_\mu(x)$, $y^R(x)$, where x stands for x^1, x^2, x^3 with $x^1 > 0$, together with the surface variables $y^R(x^2, x^3)$ with $x^1 = 0$. Their derivatives with respect to x^0, namely $A_{\mu,0}$, $y^R{}_{,0}$ are the velocities. The Lagrangian is

$$4\pi L = -\frac{1}{4}\int_{x^1>0} K F_{\mu\nu}F^{\mu\nu}\,\mathrm{d}^3x - \omega\int_{x^1=0} M\,\mathrm{d}x^2\,\mathrm{d}x^3 \tag{20}$$

expressed in terms of these co-ordinates and velocities.

To introduce the momenta we vary the velocities in L, which gives

$$4\pi\delta L = \int K\{F^{\mu 0}\delta A_{\mu,0} + F_\mu{}^\alpha F^{\mu 0} y_{R,\alpha}\,\delta y^R{}_{,0}\}\,\mathrm{d}^3x - \omega\int M c^{\alpha 0} y_{R,\alpha}\,\delta y^R{}_{,0}\,\mathrm{d}x^2\,\mathrm{d}x^3.$$

We set

$$\delta L = \int (B^\mu \delta A_{\mu,0} + w_R\,\delta y^R{}_{,0})\,\mathrm{d}^3x + \int W_R\,\delta y^R{}_{,0}\,\mathrm{d}x^2\,\mathrm{d}x^3, \tag{21}$$

so that B^μ and w_R are the momenta conjugate to A_μ and y^R in the outside space and W_R is the momentum conjugate to y^R on the surface of the electron. B^μ and w_R are functions of x^1, x^2, x^3 with $x^1 > 0$ and W_R is a function of x^2 and x^3 only. By comparing the two expressions for δL, we find

$$4\pi B^\mu = KF^{\mu 0}, \tag{22}$$

$$4\pi w_R = KF_\mu{}^\alpha F^{\mu 0} y_{R,\alpha}, \tag{23}$$

$$4\pi W_R = -\omega M c^{\alpha 0} y_{R,\alpha}. \tag{24}$$

We must eliminate the velocities from these equations to get the primary constraints of the Hamiltonian formalism. So far as concerns the outside space $x^1 > 0$, these constraints are the usual ones for the Maxwell field referred to curvilinear co-ordinates x^1, x^2, x^3, namely

$$B^0 = 0, \tag{25}$$

$$w_R y^R{}_{,s} - B^r F_{rs} = 0. \tag{26}$$

There is also a secondary constraint

$$B^r{}_{,r} = 0, \tag{27}$$

which follows from the field equations. From (24)

$$4\pi W_R y^R{}_{,\mu} = -\omega M c^{\alpha 0}(g_{\alpha\mu} - g^0_\alpha g^0_\mu). \tag{28}$$

The right-hand side vanishes for $\mu = 2$ or 3, since c^{ab} is the reciprocal matrix to g_{ab}. Thus

$$W_R y^R{}_{,2} = 0, \quad W_R y^R{}_{,3} = 0, \tag{29}$$

meaning that the tangential components of W along the surface of the electron at any time are zero.

The constraints (25), (26), (27), (29) are all first-class, according to the definition (Dirac 1958). They are associated with the arbitrariness in the gauge and in the system of curvilinear co-ordinates x^1, x^2, x^3.

The Hamiltonian is

$$H = \int (B^\mu A_{\mu,0} + w_R y^R{}_{,0})\,\mathrm{d}^3x + \int W_R y^R{}_{,0}\,\mathrm{d}x^2\,\mathrm{d}x^3 - L$$

expressed in terms of the dynamical co-ordinates and momenta. It is the sum of two parts, a volume integral H_o over the outside space and a surface integral H_s. The volume integral is

$$\begin{aligned} 4\pi H_o &= \int K\{F^{r0}A_{r,0} + F_r{}^s F^{r0}(g_{\alpha 0} - g^0_\alpha) + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\}\,\mathrm{d}^3x \\ &= \int K\{F^{r0}A_{0,r} - F_r{}^0 F^{r0} + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\}\,\mathrm{d}^3x. \end{aligned} \tag{30}$$

The term $\int KF^{r0}A_{0,r}\,\mathrm{d}^3x$ may be transformed to $-4\pi\int B^r{}_{,r}A_0\,\mathrm{d}^3x$ with the help of the boundary condition $A_0 = 0$ on the surface of the electron, and it then vanishes, from (27). To express the rest of (30) in terms of the dynamical co-ordinates and momenta, introduce e^{rs}, the reciprocal matrix to g_{rs}. It satisfies

$$e^{rs} = g^{rs} - g^{r0}g^{s0},$$

analogous to (11) with $g^{00} = 1$. We now have

$$e^{rs}e^{tu}F_{rt}F_{su} = (g^{\mu\alpha} - g^{\mu 0}g^{\alpha 0})(g^{\nu\beta} - g^{\nu 0}g^{\beta 0})F_{\mu\nu}F_{\alpha\beta}$$
$$= F_{\mu\nu}F^{\mu\nu} - 2F^{0}_{\tau}F^{\tau 0},$$

leading to
$$4\pi H_o = \int \{-8\pi^2 K^{-1} g_{rs} B^r B^s + \tfrac{1}{4} K e^{rs} e^{tu} F_{rt} F_{su}\} \mathrm{d}^3x. \tag{31}$$

It is the usual Hamiltonian for the Maxwell field, referred to the curvilinear co-ordinates x^1, x^2, x^3.

The surface part of the Hamiltonian is

$$4\pi H_s = \int (4\pi W_R y^R{}_{,0} + \omega M)\, \mathrm{d}x^2 \mathrm{d}x^3$$
$$= \omega \int M c^{00}\, \mathrm{d}x^2 \mathrm{d}x^3 \tag{32}$$

with the help of (28) with $\mu = 0$. Now

$$M^2 c^{00} = g_{22}g_{33} - g_{23}{}^2 \tag{33}$$

and is independent of the velocities, but Mc^{00} does depend on the velocities, so some further work is needed. From (24)

$$16\pi^2 W_R W_R = \omega^2 M^2 c^{a0} c^{b0} (g^0_a g^0_b - g_{ab})$$
$$= \omega^2 M^2 \{(c^{00})^2 - c^{00}\}.$$

Hence
$$4\pi H_s = \int \{16\pi^2 W_R W_R + \omega^2 M^2 c^{00}\}^{\frac{1}{2}}\, \mathrm{d}x^2 \mathrm{d}x^3$$
$$= \int \{16\pi^2 W_R W_R + \omega^2 (g_{22}g_{33} - g_{23}{}^2)\}^{\frac{1}{2}}\, \mathrm{d}x^2 \mathrm{d}x^3, \tag{34}$$

which gives H_s in terms of the dynamical co-ordinates and momenta.

We may remove from H_s the tangential part of W_R, which vanishes from (29). From

$$e^{rs} y_{R,r} y_{S,s} = -\delta_{RS}$$

we get
$$W_R W_R = -e^{rs} W_R y_{R,r} W_S y_{S,s} = -e^{11} (W_R y_{R,1})^2.$$

Also
$$-K^2 e^{11} = g_{22}g_{33} - g_{23}{}^2,$$

analogous to (33). Thus we get

$$4\pi H_s = \int \{(4\pi W_R y_{R,1}/K)^2 + \omega^2\}^{\frac{1}{2}} (g_{22}g_{33} - g_{23}{}^2)^{\frac{1}{2}}\, \mathrm{d}x^2 \mathrm{d}x^3 \tag{35}$$

as an alternative form for H_s.

The total Hamiltonian, given by (31) plus (34) or (35), is positive definite, so there are no runaway motions for the electron in this theory.

A specialized Hamiltonian

For dealing with an electron whose centre is at rest, or nearly at rest, it is convenient to work with polar co-ordinates r, θ, ϕ whose origin is at some fixed point inside the electron. Thus we take

$$x^1 = r - \rho, \quad x^2 = \theta, \quad x^3 = \phi, \tag{36}$$

where ρ is some function of θ, ϕ and t such that $r = \rho$ is the equation to the surface of the electron. The quantities $\rho(\theta, \phi)$ for a given t now appear as dynamical co-ordinates. All the previous dynamical co-ordinates $y_R(x^1, x^2, x^3)$ become functions of $\rho(\theta, \phi)$ through the equations

$$\left.\begin{aligned} y_1 &= (\rho + x^1)\cos\theta, \\ y_2 &= (\rho + x^1)\sin\theta\cos\phi, \\ y_3 &= (\rho + x^1)\sin\theta\sin\phi. \end{aligned}\right\} \tag{37}$$

The fixing of the x system of co-ordinates in this way implies bringing into the Hamiltonian theory some second-class constraints, which reduce the number of effective degrees of freedom. Out of all the y_R degrees of freedom, the only effective ones that survive are those associated with the dynamical co-ordinates $\rho(\theta, \phi)$. Their conjugate momenta, $P(\theta, \phi)$ say, must satisfy

$$[\rho(\theta, \phi),\ P(\theta', \phi')] = \delta(\theta - \theta')\,\delta(\phi - \phi'). \tag{38}$$

These relations lead to

$$\begin{aligned} [y_R(\theta, \phi),\ P(\theta', \phi')] &= (\partial y_R/\partial\rho)\,\delta(\theta - \theta')\,\delta(\phi - \phi') \\ &= y_{R,1}\,\delta(\theta - \theta')\,\delta(\phi - \phi'), \end{aligned}$$

which show that we may take $$P = W_R y^R{}_{,1}. \tag{39}$$

If one made a transformation of the co-ordinates x^2, x^3, then ρ would be a scalar and P would be a scalar density. Thus P/K would be a scalar.

With the co-ordinates (37) we find

$$\left.\begin{aligned} K &= \rho^2\sin\theta, \\ g_{22}g_{33} - g_{23}{}^2 &= \rho^2\{\rho^2\sin^2\theta + \rho_{,2}{}^2\sin^2\theta + \rho_{,3}{}^2\}. \end{aligned}\right\} \tag{40}$$

Thus the surface Hamiltonian (35) becomes

$$4\pi H_s = \int\{(4\pi P/\rho\sin\theta)^2 + \omega^2\rho^2\}^{\frac{1}{2}}\{\rho^2\sin^2\theta + \rho_{,2}{}^2\sin^2\theta + \rho_{,3}{}^2\}^{\frac{1}{2}}\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{41}$$

The spherically symmetric Hamiltonian

For a spherically symmetric motion ρ and P/K are independent of θ and ϕ. The equations that express this independence are second-class constraints, which further reduce the number of effective degrees of freedom. There is only one effective dynamical co-ordinate ρ left. Its conjugate momentum, η say, must be defined by the strong equation

$$\eta = \int P\,\mathrm{d}\theta\,\mathrm{d}\phi,$$

in order that (38) may lead to the correct P.b. relation

$$[\rho(\theta,\phi),\eta] = \int \delta(\theta-\theta')\,\delta(\phi-\phi')\,\mathrm{d}\theta'\,\mathrm{d}\phi' = 1.$$

With the help of (40) and the constraints which express that ρ and P/K are independent of θ and ϕ, we can deduce the weak equation

$$\eta = (P/K)\int K\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= 4\pi(P/K)\,\rho^2 = 4\pi P/\sin\theta.$$

Substituting in (41), we get H_s expressed in terms of the effective dynamical variables ρ and η,

$$4\pi H_s = \int (\eta^2+\omega^2\rho^4)^{\frac{1}{2}}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$$
$$= 4\pi(\eta^2+\omega^2\rho^4)^{\frac{1}{2}}.$$

To get the total Hamiltonian we must add to H_s the energy of the Coulomb field, which gives

$$H = (\eta^2+\omega^2\rho^4)^{\frac{1}{2}} + e^2/2\rho, \tag{42}$$

in agreement with (16) when $\eta = 0$.

Quantization of the spherically symmetric motion

The classical Hamiltonian (42) suggests the wave equation

$$i\hbar\,\partial\psi/\partial t = \{(\eta^2+\omega^2\rho^4)^{\frac{1}{2}} + e^2/2\rho\}\,\psi, \tag{43}$$

in which the wave function ψ is a function of ρ and t, and η means $-i\hbar\,\partial/\partial\rho$. However, there is some ambiguity, because the operator $(\eta^2+\omega^2\rho^4)^{\frac{1}{2}}$ may be replaced by another operator with the same classical analogue, for example,

$$\{(\eta+i\omega\rho^2)\,(\eta-i\omega\rho^2)\}^{\frac{1}{2}} \quad \text{or} \quad (\eta+i\omega\rho^2)^{\frac{1}{2}}\,(\eta-i\omega\rho^2)^{\frac{1}{2}}.$$

The classical theory developed here cannot decide between these possibilities.

With this uncertainty preventing an accurate quantum theory, we can still get a rough quantum theory by working with the Bohr–Sommerfeld method of quantization. This gives for the one-quantum state of radial oscillation the quantum condition

$$h = 2\int \eta\,\mathrm{d}\rho, \tag{44}$$

the integral being taken from the minimum to the maximum value for ρ, so as to correspond to half an oscillation. The connexion between η and ρ is given by (42) with H treated as a constant, so

$$\eta^2 = (H-e^2/2\rho)^2 - \omega^2\rho^4$$
$$= (e^2/4a)^2\{(k-2/x)^2 - x^4\}$$

from (15), with $x = \rho/a$ and $\qquad k = 4aH/e^2$. (45)

Thus (44) becomes $\qquad 2h/e^2 = \int\{(k-2/x)^2 - x^4\}^{\frac{1}{2}}\,\mathrm{d}x.$

For large values of k the integral is approximately

$$\int_0^{k^{\frac{1}{2}}} (k^2 - x^4)^{\frac{1}{2}} \, dx, \tag{46}$$

which goes over into $$k^{\frac{3}{2}} \int_0^1 (1 - y^4)^{\frac{1}{2}} \, dy = 0{\cdot}87 k^{\frac{3}{2}}.$$

Thus leads to $$2h/e^2 = 0{\cdot}87 k^{\frac{3}{2}}$$

giving $$k = 158. \tag{47}$$

One can easily check that this value for k is large enough for the approximation (46) to be valid.

Combining (47), (45) and (17) we get

$$H = \tfrac{1}{3} k m = 53 m.$$

So the present theory leads to a mass for the muon about a quarter of the observed mass.

The present theory can be only a rough one, because it does not give any spin to the electron. It serves to indicate how one can look upon the muon as an electron excited by radial oscillations. It gives to the muon the same spherical symmetry as the electron, so that the muon cannot disintegrate into an electron and a photon, and would have only a very small probability of disintegrating into an electron and several photons. To improve the theory and bring in the spin, one would presumably have to replace the square root in the Hamiltonian by some rational function involving spin variables.

References

Dirac, P. A. M. 1958 *Proc. Roy. Soc.* A, **246**, 326.
Lees, A. 1939 *Phil. Mag.* **28**, 385.