DO GRAVITATIONAL FIELDS PLAY AN ESSENTIAL PART IN THE STRUCTURE OF THE ELEMENTARY PARTICLES OF MATTER?

BY

A. EINSTEIN

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NEITHER the Newtonian nor the relativistic theory of gravitation has so far led to any advance in the theory of the constitution of matter. In view of this fact it will be shown in the following pages that there are reasons for thinking that the elementary formations which go to make up the atom are held together by gravitational forces.

§ 1. Defects of the Present View

Great pains have been taken to elaborate a theory which will account for the equilibrium of the electricity constituting the electron. G. Mie, in particular, has devoted deep researches to this question. His theory, which has found considerable support among theoretical physicists, is based mainly on the introduction into the energy-tensor of supplementary terms depending on the components of the electro-dynamic potential, in addition to the energy terms of the Maxwell-Lorentz theory. These new terms, which in outside space are unimportant, are nevertheless effective in the interior of the electrons in maintaining equilibrium against the electric forces of repulsion. In spite of the beauty of the formal structure of this theory, as erected by Mie, Hilbert, and Weyl, its physical results have hitherto been unsatisfactory. On the one hand the multiplicity of possibilities is discouraging, and on the other hand those additional terms have not as yet allowed themselves to be framed in such a simple form that the solution could be satisfactory.

So far the general theory of relativity has made no change in this state of the question. If we for the moment disregard the additional cosmological term, the field equations take the form

$$G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G = -\kappa T_{\mu\nu} \quad . \quad . \quad . \quad (1)$$

where $G_{\mu\nu}$ denotes the contracted Riemann tensor of curvature, G the scalar of curvature formed by repeated contraction, and $T_{\mu\nu}$ the energy-tensor of "matter." The assumption that the $T_{\mu\nu}$ do *not* depend on the derivatives of the $g_{\mu\nu}$ is in keeping with the historical development of these equations. For these quantities are, of course, the energy-components in the sense of the special theory of relativity, in which variable $g_{\mu\nu}$ do not occur. The second term on the left-hand side of the equation is so chosen that the divergence of the left-hand side of (1) vanishes identically, so that taking the divergence of (1), we obtain the equation

$$\frac{\partial \mathfrak{T}_\mu^\sigma}{\partial x_\sigma} + \tfrac{1}{2}g_\mu^{\sigma\tau}\mathfrak{T}_{\sigma\tau} = 0 \quad . \quad . \quad . \quad (2)$$

which in the limiting case of the special theory of relativity gives the complete equations of conservation

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = 0.$$

Therein lies the physical foundation for the second term of the left-hand side of (1). It is by no means settled *a priori* that a limiting transition of this kind has any possible meaning. For if gravitational fields do play an essential part in the structure of the particles of matter, the transition to the limiting case of constant $g_{\mu\nu}$ would, for them, lose its justification, for indeed, with constant $g_{\mu\nu}$ there could not be any particles of matter. So if we wish to contemplate the possibility that gravitation may take part in the structure of the fields which constitute the corpuscles, we cannot regard equation (1) as confirmed.

Placing in (1) the Maxwell-Lorentz energy-components of the electromagnetic field $\phi_{\mu\nu}$,

$$T_{\mu\nu} = \tfrac{1}{4}g_{\mu\nu}\phi_{\sigma\tau}\phi^{\sigma\tau} - \phi_{\mu\sigma}\phi_{\nu\tau}g^{\sigma\tau}, \quad . \quad . \quad (3)$$

we obtain for (2), by taking the divergence, and after some reduction,*

$$\phi_{\mu\sigma}\mathfrak{J}^{\sigma} = 0 \quad . \quad . \quad . \quad . \quad (4)$$

where, for brevity, we have set

$$\frac{\partial}{\partial x_{\tau}}(\sqrt{-g}\ \phi_{\mu\nu}g^{\mu\sigma}g^{\nu\tau}) = \frac{\partial \mathfrak{f}^{\sigma\tau}}{\partial x_{\tau}} = \mathfrak{J}^{\sigma} \quad . \quad . \quad (5)$$

In the calculation we have employed the second of Maxwell's systems of equations

$$\frac{\partial \phi_{\mu\nu}}{\partial x_{\rho}} + \frac{\partial \phi_{\nu\rho}}{\partial x_{\mu}} + \frac{\partial \phi_{\rho\mu}}{\partial x_{\nu}} = 0 \quad . \quad . \quad . \quad (6)$$

We see from (4) that the current-density $\mathfrak{J}^{\sigma}$ must everywhere vanish. Therefore, by equation (1), we cannot arrive at a theory of the electron by restricting ourselves to the electromagnetic components of the Maxwell-Lorentz theory, as has long been known. Thus if we hold to (1) we are driven on to the path of Mie's theory.†

Not only the problem of matter, but the cosmological problem as well, leads to doubt as to equation (1). As I have shown in the previous paper, the general theory of relativity requires that the universe be spatially finite. But this view of the universe necessitated an extension of equations (1), with the introduction of a new universal constant λ, standing in a fixed relation to the total mass of the universe (or, respectively, to the equilibrium density of matter). This is gravely detrimental to the formal beauty of the theory.

§ 2. The Field Equations Freed of Scalars

The difficulties set forth above are removed by setting in place of field equations (1) the field equations

$$G_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}G = -\kappa T_{\mu\nu} \quad . \quad . \quad . \quad (1a)$$

where $T_{\mu\nu}$ denotes the energy-tensor of the electromagnetic field given by (3).

The formal justification for the factor $-\frac{1}{4}$ in the second

* Cf. e.g. A. Einstein, Sitzungsber. d. Preuss. Akad. d. Wiss., 1916, pp. 187, 188.

† Cf. D. Hilbert, Göttinger Nachr., 20 Nov., 1915.

term of this equation lies in its causing the scalar of the left-hand side,

$$g^{\mu\nu}(G_{\mu\nu} - \tfrac{1}{4}g_{\mu\nu}G),$$

to vanish identically, as the scalar $g^{\nu\mu}T_{\mu\nu}$ of the right-hand side does by reason of (3). If we had reasoned on the basis of equations (1) instead of (1a), we should, on the contrary, have obtained the condition $G = 0$, which would have to hold good everywhere for the $g_{\mu\nu}$, independently of the electric field. It is clear that the system of equations [(1a), (3)] is a consequence of the system [(1), (3)], but not conversely.

We might at first sight feel doubtful whether (1a) together with (6) sufficiently define the entire field. In a generally relativistic theory we need $n - 4$ differential equations, independent of one another, for the definition of n independent variables, since in the solution, on account of the liberty of choice of the co-ordinates, four quite arbitrary functions of all co-ordinates must naturally occur. Thus to define the sixteen independent quantities $g_{\mu\nu}$ and $\phi_{\mu\nu}$ we require twelve equations, all independent of one another. But as it happens, nine of the equations (1a), and three of the equations (6) are independent of one another.

Forming the divergence of (1a), and taking into account that the divergence of $G_{\mu\nu} - \frac{1}{2}g_{\mu\nu}G$ vanishes, we obtain

$$\phi_{\sigma\alpha}J^{\alpha} + \frac{1}{4\kappa}\frac{\partial G}{\partial x_{\sigma}} = 0 \quad . \quad . \quad . \quad (4a)$$

From this we recognize first of all that the scalar of curvature G in the four-dimensional domains in which the density of electricity vanishes, is constant. If we assume that all these parts of space are connected, and therefore that the density of electricity differs from zero only in separate "world-threads," then the scalar of curvature, everywhere outside these world-threads, possesses a constant value G_0. But equation (4a) also allows an important conclusion as to the behaviour of G within the domains having a density of electricity other than zero. If, as is customary, we regard electricity as a moving density of charge, by setting

$$J^{\sigma} = \frac{\mathfrak{J}^{\sigma}}{\sqrt{-g}} = \rho\frac{dx_{\sigma}}{ds}, \quad . \quad . \quad . \quad (7)$$

we obtain from (4a) by inner multiplication by J^σ, on account of the antisymmetry of $\phi_{\mu\nu}$, the relation

$$\frac{\partial G}{\partial x_\sigma}\frac{dx_\sigma}{ds} = 0 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Thus the scalar of curvature is constant on every world-line of the motion of electricity. Equation (4a) can be interpreted in a graphic manner by the statement: The scalar of curvature plays the part of a negative pressure which, outside of the electric corpuscles, has a constant value G_0. In the interior of every corpuscle there subsists a negative pressure (positive $G - G_0$) the fall of which maintains the electrodynamic force in equilibrium. The minimum of pressure, or, respectively, the maximum of the scalar of curvature, does not change with time in the interior of the corpuscle.

We now write the field equations (1a) in the form

$$(G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G) + \tfrac{1}{4}g_{\mu\nu}G_0 = -\kappa\Big(T_{\mu\nu} + \frac{1}{4\kappa}g_{\mu\nu}(G - G_0)\Big) \quad (9)$$

On the other hand, we transform the equations supplied with the cosmological term as already given

$$G_{\mu\nu} - \lambda g_{\mu\nu} = -\kappa(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T).$$

Subtracting the scalar equation multiplied by $\frac{1}{2}$, we next obtain

$$(G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G) + g_{\mu\nu}\lambda = -\kappa T_{\mu\nu}.$$

Now in regions where only electrical and gravitational fields are present, the right-hand side of this equation vanishes. For such regions we obtain, by forming the scalar,

$$-G + 4\lambda = 0.$$

In such regions, therefore, the scalar of curvature is constant, so that λ may be replaced by $\frac{1}{4}G_0$. Thus we may write the earlier field equation (1) in the form

$$G_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}G + \tfrac{1}{4}g_{\mu\nu}G_0 = -\kappa T_{\mu\nu} \quad . \quad . \quad (10)$$

Comparing (9) with (10), we see that there is no difference between the new field equations and the earlier ones, except that instead of $T_{\mu\nu}$ as tensor of "gravitating mass" there now occurs $T_{\mu\nu} + \frac{1}{4\kappa} g_{\mu\nu}(G - G_0)$ which is independent of the scalar of curvature. But the new formulation has this great advantage, that the quantity λ appears in the fundamental equations as a constant of integration, and no longer as a universal constant peculiar to the fundamental law.

§ 3. On the Cosmological Question

The last result already permits the surmise that with our new formulation the universe may be regarded as spatially finite, without any necessity for an additional hypothesis. As in the preceding paper I shall again show that with a uniform distribution of matter, a spherical world is compatible with the equations.

In the first place we set

$$ds^2 = -\gamma_{ik}dx_idx_k + dx_4^2 \; (i, k = 1, 2, 3) \qquad (11)$$

Then if P_{ik} and P are, respectively, the curvature tensor of the second rank and the curvature scalar in three-dimensional space, we have

$$\begin{aligned} G_{ik} &= P_{ik} \; (i, k = 1, 2, 3) \\ G_{i4} &= G_{4i} = G_{44} = 0 \\ G &= -P \\ -g &= \gamma. \end{aligned}$$

It therefore follows for our case that

$$\begin{aligned} G_{ik} - \tfrac{1}{2}g_{ik}G &= P_{ik} - \tfrac{1}{2}\gamma_{ik}P \; (i, k = 1, 2, 3) \\ G_{44} - \tfrac{1}{2}g_{44}G &= \tfrac{1}{2}P. \end{aligned}$$

We pursue our reflexions, from this point on, in two ways. Firstly, with the support of equation (1a). Here $T_{\mu\nu}$ denotes the energy-tensor of the electro-magnetic field, arising from the electrical particles constituting matter. For this field we have everywhere

$$\mathfrak{T}_1^1 + \mathfrak{T}_2^2 + \mathfrak{T}_3^3 + \mathfrak{T}_4^4 = 0.$$

The individual $\mathfrak{T}_\mu^\nu$ are quantities which vary rapidly with position; but for our purpose we no doubt may replace them by their mean values. We therefore have to choose

$$\left.\begin{aligned} \mathfrak{T}_1^1 = \mathfrak{T}_2^2 = \mathfrak{T}_3^3 &= -\tfrac{1}{3}\mathfrak{T}_4^4 = \text{const.} \\ \mathfrak{T}_\mu^\nu &= 0 \;\; (\text{for } \mu \neq \nu), \end{aligned}\right\} \quad . \quad . \quad (12)$$

and therefore

$$T_{ik} = \frac{1}{3}\frac{\mathfrak{T}_4^4}{\sqrt{\gamma}}\gamma_{ik},\ T_{44} = \frac{\mathfrak{T}_4^4}{\sqrt{\gamma}}.$$

In consideration of what has been shown hitherto, we obtain in place of (1a)

$$P_{ik} - \frac{1}{4}\gamma_{ik}P = -\frac{1}{3}\gamma_{ik}\frac{\kappa\mathfrak{T}_4^4}{\sqrt{\gamma}} \quad . \quad . \quad . \quad (13)$$

$$\frac{1}{4}P = -\frac{\kappa\mathfrak{T}_4^4}{\sqrt{\gamma}} \quad . \quad . \quad . \quad (14)$$

The scalar of equation (13) agrees with (14). It is on this account that our fundamental equations permit the idea of a spherical universe. For from (13) and (14) follows

$$P_{ik} + \frac{4}{3}\frac{\kappa\mathfrak{T}_4^4}{\sqrt{\gamma}}\gamma_{ik} = 0 \quad . \quad . \quad . \quad (15)$$

and it is known* that this system is satisfied by a (three-dimensional) spherical universe.

But we may also base our reflexions on the equations (9). On the right-hand side of (9) stand those terms which, from the phenomenological point of view, are to be replaced by the energy-tensor of matter; that is, they are to be replaced by

$$\begin{matrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho \end{matrix}$$

where ρ denotes the mean density of matter assumed to be at rest. We thus obtain the equations

$$P_{ik} - \frac{1}{4}\gamma_{ik}P - \frac{1}{4}\gamma_{ik}G_0 = 0 \quad . \quad . \quad (16)$$

$$\frac{1}{4}P + \frac{1}{4}G_0 = -\kappa\rho \quad . \quad . \quad . \quad . \quad (17)$$

From the scalar of equation (16) and from (17) we obtain

$$G_0 = -\frac{2}{3}P = 2\kappa\rho, \quad . \quad . \quad . \quad (18)$$

and consequently from (16)

$$P_{ik} - \kappa\rho\gamma_{ik} = 0 \quad . \quad . \quad . \quad (19)$$

* Cf. H. Weyl, "Raum, Zeit, Materie," § 33.

which equation, with the exception of the expression for the co-efficient, agrees with (15). By comparison we obtain

$$\mathfrak{T}_4^4 = \tfrac{3}{4}\rho\sqrt{\gamma}. \qquad (20)$$

This equation signifies that of the energy constituting matter three-quarters is to be ascribed to the electromagnetic field, and one-quarter to the gravitational field.

§ 4. Concluding Remarks

The above reflexions show the possibility of a theoretical construction of matter out of gravitational field and electromagnetic field alone, without the introduction of hypothetical supplementary terms on the lines of Mie's theory. This possibility appears particularly promising in that it frees us from the necessity of introducing a special constant λ for the solution of the cosmological problem. On the other hand, there is a peculiar difficulty. For, if we specialize (1) for the spherically symmetrical static case we obtain one equation too few for defining the $g_{\mu\nu}$ and $\phi_{\mu\nu}$, with the result that *any spherically symmetrical distribution* of electricity appears capable of remaining in equilibrium. Thus the problem of the constitution of the elementary quanta cannot yet be solved on the immediate basis of the given field equations.