# Quantum Limitations of the Measurement of Space-Time Distances

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This article deals with the limitations which the quantized nature of microscopic systems imposes on the possibility of measuring distances between space-time events. It is proposed to use only clocks for measuring space-time distances and to avoid the use of measuring rods which are essentially macrophysical objects. The accuracy of reading a clock with a given mass is considered and examples for microphysical clocks are given. It is shown that the mass of the clock, and the uncertainty (spread) of this quantity, exceed certain values which depend on the accuracy with which the time interval is to be measured, the magnitude of this time interval (the running time of the clock) and the size of the clock. The minimum mass uncertainty of the clock is given by Heisenberg's relation; the minimum mass itself is higher by the ratio of the running time and the accuracy.

If the possibility of constructing states whose wave functions are Gaussian wave packets is admitted, the mass and the mass uncertainty of the clock differs only by logarithmic factors from the uncertainties which follow from general principles of quantum mechanics. The masses are much higher if the possibility of constructing arbitrary wave packets is not admitted.

## 1. INTRODUCTION

IN ordinary quantum mechanics the space-time point is specified by its four coordinates but no prescription is given how these coordinates are to be measured. This in turn is in conflict with the principles of the general theory of relativity according to which coordinates have no meaning independent of observation.[1] The basic measurement of general relativity is the measurement of distances between events in space-time. Such measurements make the definition of a coordinate system possible if they can be carried out without restrictions. We shall therefore examine the limitations which quantum mechanics imposes on the possibility of measuring distances between events in space-time. Only when this question is answered will it be possible to treat the problem whether the gravitational field of atomic systems and of elementary particles is observable in principle.

Before proceeding with the proposition of a clock, the possibility of the measurement of space-like distances with clocks should be pointed out. This is, in principle, quite simple in classical theory and is illustrated in Fig. 1. It applies if the distance involved is small as compared with the curvature of space. Denoting the components of the unit vector tangent to the world line by $e^i$, the components of the vector leading from point I to Event 1 become $te^i$, those of the vector from Event 1 to II are $t'e^i$. The components of the vector leading from Event 1 to Event 2 shall be denoted by $x^i$. Since I and Event 2 are on a null line, we have

$$g_{ik}(te^i+x^i)(te^k+x^k)=0, \tag{1a}$$

and similarly

$$g_{ik}(t'e^i-x^i)(t'e^k-x^k)=0. \tag{1b}$$

Multiplication of these equations by $t'$ and $t$ and addition eliminates the terms linear in $t$ or $t'$ and gives

$$g_{ik}(t^2t'+tt'^2)e^ie^k+g_{ik}(t+t')x^ix^k=0.$$

Division by $t+t'$, together with the condition that $e$ is a unit vector, i.e., that $g_{ik}e^ie^k=1$, gives

$$g_{ik}x^ix^k=-tt'; \tag{1}$$

that is, the absolute value of the distance of the two events is the geometric average of the time intervals $t$ and $t'$. If a clock of arbitrary accuracy existed and if the recoil of the light signals could be disregarded, it would be possible to measure space-like distances with arbitrary accuracy.

The function of the clock to be considered is to measure the distance between two events, which shall consist of collisions between material objects and light quanta. As is well known, and as was pointed out most clearly by von Neumann, the measurement is not completed until its result is recorded by some macroscopic object.[2] If the macroscopic object were part of the clock, no microscopic clock could exist. The way out of this difficulty is to transmit the signal of the clock to a macroscopic recorder (which can be the "final observer") which is far away from the clock, considered from the point of view of the average motion of the latter. The transmitting signal will be considered to be part of the clock, not, however, the recording apparatus. This concept forces upon us the most important and possibly decisive limitation: if the transmitting signal is to be microscopic, that is, if it is to consist of only a few quanta (actually, our signals will be light quanta), it will reach the recording equipment with certainty only if it does not spread out in every direction. In order to guarantee this, we confine ourselves to a world which has, in addition to the time-like dimension, only one

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[1] See, e.g., E. P. Wigner, in *Jubilee of Relativity Theory*, edited by A. Mercier and M. Kervaire (Birkhäuser Verlag, Basel, 1956), p. 210.

[2] J. von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer Verlag, Berlin, 1932; also Princeton University Press, Princeton, 1955), Chap. 6.

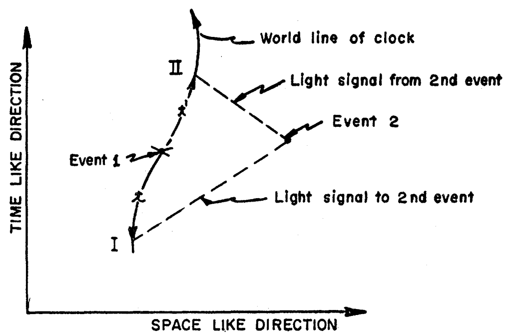


FIG. 1. Reduction of the measurement of space-like distances to the measurement of time-like distances. The absolute value of the distance between events 1 and 2 is the geometric mean of the time intervals $t$ and $t'$.

space-like dimension. The principal weakness of this assumption is that it may be questionable whether it is possible to separate clock and recorder in a world with only one space-like dimension. The action of a body in such a world does not necessarily decrease with increasing distance and the macroscopic recorder may have a substantial influence in the region where the clock is situated. This region, however, was to be freed from the effect of macroscopic bodies and it is for this purpose that we wish to construct a microscopic clock. This difficulty, serious as it may be, will be disregarded in what follows. Its significance is weakened by the circumstance that the clocks which we can construct are not wholly microscopic and the need of focusing their signals may not increase their mass too much.

## 2. CRITERIA WHICH FOLLOW FROM GENERAL PRINCIPLES OF QUANTUM MECHANICS

The present section will deal with properties of the clock which follow from general principles of quantum mechanics. They will be valid even if one adopts the most liberal attitude towards the realizability of physical instruments. Such an attitude underlies also the investigations of Bohr and Rosenfeld on the measurability of the electromagnetic field.[3] Another case in which the same general principles impose limitations on the measuring apparatus is the measurement whether the spin of a particle is parallel or antiparallel to a given direction. The condition in this case is[4] that the angular momentum of the measuring equipment show a spread of the order $h/\delta^{\frac{1}{2}}$ if the measurement is to give the correct result with a probability $1-\delta$. The limitations to be found in the present section are of a similar nature and of a similar origin.

The clock to be considered shall have an accuracy $\tau$ and be able to measure time intervals up to a maximum $T=n\tau$. Its linear dimension shall be not larger than $l$. The questions which we wish to answer are (a) what is the minimum mass of the clock; (b) what is the minimum mass (or energy) uncertainty of the clock which satisfies the foregoing specifications. The "events" are arrivals of light quanta; the clock is expected to measure the distance between such events. It is this kind of measurement which is necessary to determine the curvature of space.

Let us consider first the first two conditions. It follows from these that, in the course of the time $T$, the quantum mechanical state of the clock must go through $n$ orthogonal vectors or, equivalently, that the wave function $\varphi$ of the clock shall be the superposition of at least $n$ stationary states $\psi_1, \psi_2, \cdots, \psi_n$:

$$\varphi(t)=\sum_1^n a_k\psi_k e^{-i\omega_k t}. \tag{2}$$

The energy values of the stationary states were denoted by $\hbar\omega_k$. The wave function $\varphi$ will indeed go through $n$ orthogonal states at times $\tau, 2\tau, \cdots, n\tau$ if $a_k=n^{-\frac{1}{2}}$ and $\omega_k=\omega_0+2\pi k/n\tau=\omega_0+2\pi k/T$. This means an uncertainty in the energy of the order

$$\epsilon=\hbar(\omega_n-\omega_0)=2\pi\hbar/\tau, \tag{2a}$$

and corresponds to Heisenberg's uncertainty principle. Conversely, it can be shown that $\varphi$ cannot go through $n$ orthogonal states during a time interval $T$ unless its energy uncertainty is of the order $nh/T=h/\tau$.

A further condition for the clock can be derived from the postulate that it shall show the proper time even after having been read once. This condition is necessary if the clock is to be used in the way outlined elsewhere for the measurement of the curvature but is, also apart from this use, a natural postulate. It follows from it that the clock must not be deflected too much from its original world line by being read. This requirement is also the basis of Schrödinger's observation[5]; it implies that the mass of the clock must exceed a certain amount.

The reading of the clock is connected with the emission of a light signal of duration $\tau$ and this imparts to the clock an indeterminate momentum $\hbar/c\tau$. This momentum[6] would be even greater if a particle of nonzero rest mass were used as a signal. As a result of the emission of the light signal the velocity of the clock acquires a spread of the amount $\hbar/Mc\tau$, where $M$ is the mass of the clock. After a further time interval $T_2$, it may be at a distance $\hbar T_2/Mc\tau$ from the point where it would have been without having been read. Hence the actual distance between the two points in space time, at the first of which the clock read $T_1$ less than at the time of the emission of the signal, at the second of

[3] N. Bohr and L. Rosenfeld, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. 12, No. 8 (1933).

[4] E. Wigner, Z. Physik 133, 101 (1952).

[5] E. Schrödinger, Preuss. Akad. Wiss. Berlin Ber. 12, 238 (1931).

[6] If this momentum uncertainty is compensated by the emission of another quantum of equal momentum uncertainty into the opposite direction, the position of the clock will be displaced by an indeterminate amount, due to the uncertainty of the center-of-mass of the two light quanta. For a discussion of this point, see for instance, D. Bohm, *Quantum Theory* (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1951), Chaps. 6, 7, and 22.

which it reads $T_2$ more than at the time of the emission of the light signal, is (see Fig. 2)

$$[(T_1+T_2')^2-(\hbar T_2/Mc^2\tau)^2]^{\frac{1}{2}} \approx T_1+T_2'-\frac{(\hbar T_2/Mc^2\tau)^2}{2(T_1+T_2')}, \quad (3a)$$

where

$$T_2'=[T_2^2+(\hbar T_2/Mc^2\tau)^2]^{\frac{1}{2}}\approx T_2+\tfrac{1}{2}T_2(\hbar/Mc^2\tau)^2. \quad (3b)$$

Hence the actual distance (3a) differs from the time difference $T_1+T_2$ shown by the clock, in the approximation considered, by

$$-\frac{T_1T_2}{2(T_1+T_2)}\left(\frac{\hbar}{Mc^2\tau}\right)^2. \quad (3)$$

The inaccuracy of the clock will be within the limit $\tau$ if (3) is less than $\tau$. If one considers the first factor to be of the order of magnitude $T$, this gives

$$M>(\hbar/c^2\tau)(T/\tau)^{\frac{1}{2}}. \quad (4)$$

An even higher limit is obtained if one stipulates that the position of the clock shows so little spread that the time at which a light quantum strikes it shall be predetermined within a period $\tau$. This condition can also be stated as the requirement that the position of the clock shall not introduce a statistical element into the measurement of time. It requires that the spread $\lambda$ in the position of the clock shall be, throughout the time interval $T$,

$$\lambda<c\tau. \quad (5a)$$

Again, the use of a signal with nonzero rest mass would give a more rigorous limit. The spread in the velocity

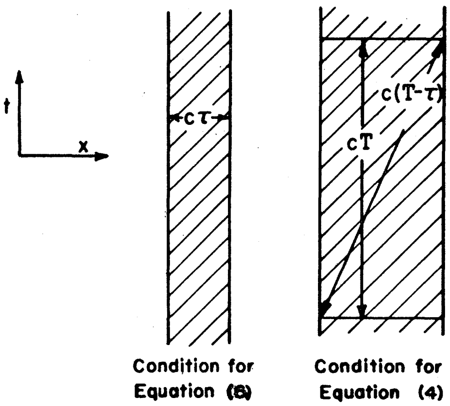


FIG. 3. The left side of the figure illustrates the condition that the clock shall remain, in spite of the recoil illustrated in Fig. 2, within the distance $c\tau$ from the point where it would have arrived had it not emitted a light signal. The right side of the figure corresponds to the requirement that its reading shall differ by less than $\tau$ from the correct distance of two points through which it passes. The corresponding minimum masses are given by (6) and (4).

of the clock is of the order $\hbar/M\lambda$, so that the uncertainty in position, after a time interval $T$, becomes

$$\lambda+\hbar T/M\lambda, \quad (5b)$$

and this should still be smaller than $c\tau$. For given $M$, (5b) assumes its minimum for

$$\lambda=(\hbar T/M)^{\frac{1}{2}}, \quad (5)$$

and this is also the order of magnitude of the expression (5b) itself. This will be smaller than $c\tau$ if

$$M>(\hbar/c^2\tau)(T/\tau). \quad (6)$$

Note that the uncertainty in the momentum of the light signal, $\hbar/c\tau$, is well below $Mc$, no matter whether (4) or (6) is adopted. Hence the use of the approximate expressions in (3a) and (3b) was justified.

The difference between the expressions (4) and (6) can be formulated also in the following way. The more stringent requirement (6) demands that the wave packet of the center-of-mass of the clock be confined, throughout the time interval $T$, to a region of the size $c\tau$. The less stringent requirement (4) guarantees only that the wave packet is sufficiently confined for the space-time distance, between any points of the wave packets an interval $T$ removed from each other in space time, to be equal $T$ within an accuracy $\tau$. This allows a spatial spread of the wave packets of the order $c\tau(T/\tau)^{\frac{1}{2}}$, that is, a much larger spread than $c\tau$ (see Fig. 3).

Neither of the two conditions, (2a) or (6), makes use of the requirement that the physical dimensions of the clock shall be limited. Nevertheless, the energy levels, of the quantum mechanical system which we are considering to be the clock, are extremely closely spaced: if the uncertainty in the energy is to be of the order (2a), the spacing of the energy levels is $\hbar/T$. One

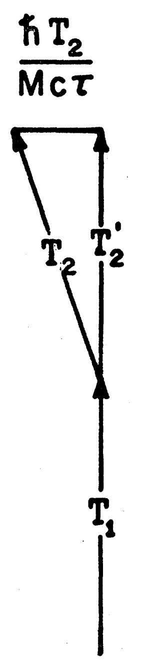


FIG. 2. The emission of a light signal confined to a time interval $\tau$ produces a recoil in the motion of the clock. As a result, the clock will proceed along the arrow marked $T_2$. The true distance between the tip of this arrow and the original position of the clock, at the bottom of the figure, differs from the indication $T_1+T_2$ of the clock.

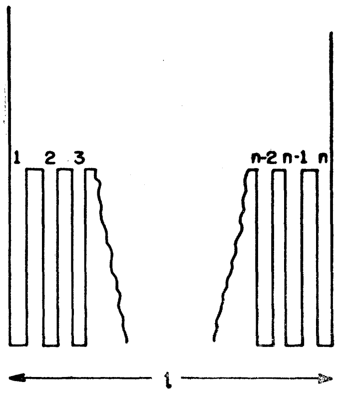


FIG. 4. Schematic picture of a potential which confines a particle to a region of space of very small extension $l$ and has a great number of very closely spaced energy levels. A particle in such a potential might consitute a clock of very small size, great accuracy, and long running time. Its realizability is open to question.

usually associates a small spacing of the energy levels with a loosely bound system, such as a very soft oscillator. This, however, need not be the case if an arbitrary form of the potential energy is considered admissible; a potential of the form shown in Fig. 4 will have $n$ closely spaced levels. One can adjust the constants of the system in such a way that the spacing becomes $\hbar/T$ and a wave function of the form (2), again with $a_k=n^{-\frac{1}{2}}$, will have the property that the distance of the particles changes, in the time element $\tau$, from one trough to the other. In this way, the distance of the particles can be considered as the pointer of the clock. Furthermore, one can adjust the constants of the system in such a way that a collision with a light quantum have hardly any probability of changing the state of the clock. However, in order to read such a clock in any simple way, one would have to use a light quantum of length $l/n$ and hence energy uncertainty $\epsilon=n\hbar c/l$, i.e., for $l=c\tau$ an $n$ times higher uncertainty than Heisenberg's principle demands. This, and similar other attempts to construct a clock of small extension, indicate that the requirement of a small size does impose further conditions on the properties, in particular the energy spread, of the clock-and-signal system, even though we were unable to derive these solely from the general principles of quantum theory. We shall now proceed to the description of a clock which consists only of noninteracting particles and the realizability of which is, in principle, hardly open to question.

### 3. EXAMPLE OF A SIMPLE MICROSCOPIC CLOCK

We have considered several types of clocks with a running time $T$ and accuracy $\tau$. These include:

(a) An ensemble of atoms in an excited state. The time elapsed is obtained from the fraction of the atoms which have decayed.

(b) An ensemble of oscillators of frequency $1/2T$, originally all in the state $2^{-\frac{1}{2}}(\psi_0+\psi_1)$ where $\psi_0$ and $\psi_1$ are, respectively, normal and first excited states. The time elapsed is obtained as the transition probability into the state $2^{-\frac{1}{2}}(\psi_0+\psi_1)$. In order to determine this transition probability with a high accuracy, the number of oscillators must be very large. Since the transition probability varies with time as $\cos^2(\pi t/2T)$, the measurement of the transition probability gives a measure of $t$.

(c) An ensemble of oscillators with frequencies $1/2T$, $1/T$, $2/T$, $4/T$, $\cdots$, $1/\tau$ in similar states as the oscillators in example (b). The time elapsed is again obtained by measuring the transition probabilities into the original state.

(d) A single oscillator of frequency $1/2T$ in the state given by (2) with $a_k=n^{-\frac{1}{2}}$. The time is obtained by measuring the position of the oscillating particle.

The discussion of all these examples gave conditions equivalent with (2a) and (6) if the necessity of reading the clock[5] is taken into account. However, the realizability and the possibility of "reading" of all these devices is open to some doubt and their physical dimensions cannot be easily obtained from general principles. The last example, a single oscillator, is most nearly free from these objections but its analysis showed that the potential between the oscillating particles played only a subordinate role. We prefer, therefore, to analyze the motion of two (or, as we shall see, actually three) particles with respect to each other and to measure the time elapsed by measuring ratios of their distances. In this way we free ourselves from the question of realizability. The discussion of (d) leads to the same conclusions as the following discussion. While the requirements (2a) and (6), since they follow from accepted principles, are certainly valid but may have to be supplemented, the requirements to be obtained below are certainly sufficient but may have to be relaxed if a more clever device for the measurement of time intervals is found.

It was noted before that, in example (d), the time is obtained as the distance of the oscillating particles from each other. A distance, however, is not a relativistically invariant concept and can, therefore, not be transmitted by a signal. For this reason, the clock must show the time by means of a quantity which is, at least in the approximation which is to be used, relativistically invariant. Such a quantity is the ratio of two distances and Fig. 5 shows the principle of the measurement proposed under the neglect of quantum effects. There are three material bodies, two of which are at rest with respect to each other, while the middle one moves toward the body at left. The time is indicated as the ratio of the distances between 1 and 2 and between 1 and 3. It is transmitted by three light quanta which travel together toward the clock but each of which is reflected by another one of the particles. No matter where the final observer is situated, and in what state of motion he is, he will obtain the same ratio between the time intervals of the passages of the three light quanta as long as these time intervals are short as compared with the radius of curvature of space-time. The same condition is necessary throughout the travel of the light quanta from clock to final observer if the intervening space is not to "warp" the signal.

Let us consider the time to be the proper time along the line midway between particles 1 and 3. If all the light signals were infinitely short, the equation of the incoming quanta which would strike the clock at $t_0$ would be

$$x=c(t-t_0), \tag{7}$$

if we assume the world to be flat and use a coordinate system at rest with respect to particles 1 and 3, its origin where the world line of particle 2 intersects the line midway between particles 1 and 3. The world lines of the three reflected light quanta then become

$$x=-c(t-t_0)-2l, \tag{7a}$$

$$x=-c\left(t-\frac{c+v}{c-v}t_0\right), \tag{7b}$$

$$x=-c(t-t_0)+2l. \tag{7c}$$

The distance of particles 1 and 3 is $2l$ in the coordinate system in which both are at rest, the velocity of particle 2 in this coordinate system is $v$ (negative in the figure). If the final observer measures the time intervals between the passages of the first and second, and between the passages of the first and third light quanta, he finds for the ratio of these time intervals

$$r=\tfrac{1}{2}+\left(\frac{v}{c-v}\right)\frac{ct_0}{2l}. \tag{8}$$

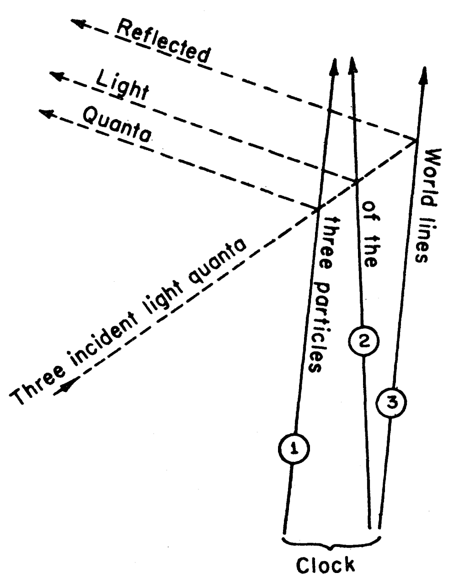


FIG. 5. Example of a realizable clock. The lines 1, 2, and 3 are world lines of material particles; the ratio of the distances 1–2 and 1–3 is the pointer of the clock. The position of the pointer is read by three light quanta, traveling together toward the clock, but each of which is reflected by a different one of the material particles. The ratio f the distances between the light quanta can be ascertained far from the clock.

Hence he can express the time $t_0$ indicated by the clock in terms of the ratio he found as

$$t_0=\left(\frac{c-v}{v}\right)\frac{l}{c}(2r-1). \tag{8a}$$

As was mentioned before, the ratio $r$ is independent of the relative motion of final observer and clock and will be preserved in a curved space also if the radius of curvature is large as compared with the size of the clock. If the physical dimension of the clock is to remain $2l$, its running time will have to be restricted to

$$T=2l/v. \tag{9}$$

It remains now to obtain the minimum mass $M$, and the uncertainty in this quantity $\epsilon/c^2$, which will permit the measurement to be carried out with an accuracy $\tau$. The ratio $r$ between the time intervals of the light signals varies from 0 to 1 during the period $T$. It must, therefore, be measurable with an accuracy $1/n=\tau/T$. This shows that the length of the light trains representing the quanta must not exceed about $1/n$ of their distance. In the coordinate system in which the clock is, on the average, at rest, and in which $\epsilon$ is measured, this amounts to a length of the order $2l/n$ and hence an energy uncertainty of the signal

$$\epsilon_s=nc\hbar/2l. \tag{10}$$

All our estimates give only orders of magnitude. Similarly, the spread of the wave packets of the three particles must remain, throughout the running time of the clock, of the order $l/n$. This gives a momentum uncertainty of the order $\hbar n/l$ which is of the same order of magnitude as the additional momentum uncertainty caused by the recoil of the light quanta. The latter can be disregarded, therefore, as long as we are not interested in numerical factors. The velocity uncertainty of the particles therefore becomes $\hbar n/Ml$, where $M$ is the mass of the particles and the order of magnitude of the mass of the clock. In order to assure that the velocity uncertainty does not cause, within the time interval $T$, a spread of the wave packet which is in excess of $l/n$, we must have

$$M>\hbar n^2T/l^2=\hbar T^3/l^2\tau^2. \tag{11}$$

The corresponding $Mc^2$ is larger than the energy of the light quanta as long as $l<cT$, which is a necessary condition. Hence, the mass of the signaling device, given by (10), can be neglected as compared with the mass of the clock proper and the latter is given by (11). It differs from the estimate (6) by the factor $c^2T^2/l^2$. Since (6) corresponds to $l\approx cT/n$, the ratio between the two quantities is $n^2$.

The momentum uncertainty $\hbar n/l$ of the particles corresponds to a velocity spread

$$(1/M)(\hbar n/l)<l/nT,$$

which is smaller than the average velocity of particle 2, given by (9). Hence the energy spread of the clock proper is

$$\epsilon_c = v\hbar n/l = 2\hbar n/T = 2\hbar/\tau. \qquad (11a)$$

This is the same value (2a) which was obtained in the preceding section and corresponds to Heisenberg's relation. It is smaller than the energy uncertainty (10) of the signal by the factor $l/cT$. It may be of interest to note that the signal is within the area of the clock only for the period $l/c$ so that it contributes, on the average, within the period $T$ and while it is in the area of the clock, only the amount $\epsilon_c$ to the energy uncertainty. Nevertheless, the total energy uncertainty of clock plus signals is larger than is demanded by Heisenberg's principle (2a) and is given by (10).

The fact that a realizable clock is subject to more severe limitations than could be obtained on the basis of general principles was foreshadowed already in the discussion of the arrangement of Fig. 4. The idealization of this arrangement related only to the clock proper and made it possible to reduce only the mass of the clock proper. Since it did not relate to the signaling system, the total energy uncertainty already corresponded to (10).

### 4. COMPOSITE MICROSCOPIC CLOCK

The large discrepancy between (11) and (6), and between (10) and (2a), suggests the construction of a more complicated clock which, with the same mass and with the same mass uncertainty as the clock of the preceding example, can measure time more accurately or is able to run for a longer period. The principle of the clock to be described next is similar to that of example (c)— and of actual clocks. It will have, instead of the single pointer, several pointers: one of these distinguishes time elements of the order $\tau$ but has a period $n_1\tau$, where $n_1 < n$, so that it cannot distinguish the times $t$ and $t + n_1\tau$, $t + 2n_1\tau$, etc. In order to distinguish between these times, one has a second pointer with an accuracy of the order $n_1\tau$. Even this may be periodic, with a period $n_2\tau > n_1\tau$ so that the two devices together may not be able to distinguish between $t$ and $t + n_2\tau$. However, together, they can measure the time with an accuracy $\tau$, over any time interval of length $n_2\tau$. There may be, then, a third pointer with an even longer period but with an accuracy of the order of only $n_2\tau$, and so on. The last pointer must have a period $T$ or $2T$ and an accuracy of the period of the preceding pointer. The total number of pointers will be denoted by $k$; they play the roles of the pointers of ordinary watches. Naturally, the reading of a clock of this construction is more complicated than the reading of the clock of the preceding section because each pointer has to be read separately. It is clear, nevertheless, that the use of several pointers may result in a substantial saving in mass and mass uncertainty.

It would be natural to "lock" the different pointers to each other, that is, to govern the motion of the second pointer by the first pointer, and so on. For this, we found no simple device. It should be noted, furthermore, that while "accuracy $\tau$" allows the possibility of an error of the order of magnitude $\tau$ in the reading of the first pointer, with a probability of about $\frac{1}{2}$, it does not allow a similar error in the second pointer. Such an error would lead to an error of the order $n_1\tau$ in the time and is, therefore, not permissible. Hence the probability of a false reading of the second pointer must be $1/n_1$ or less. Similarly, the probability of an error in the indication of the third pointer must be $1/n_2$ or less. The total mass, and the mass uncertainty, will be the sums of the corresponding quantities for all pointers and all signals.

Every pointer will be a particle moving back and forth between the particles 1 and 3 of Fig. 5. When a particle reaches one of these particles, it will suffer an elastic collision and to be returned. Particles 1 and 3 will attract each other sufficiently to compensate for the average momentum transferred to them; their mass can be the sum of the masses of all the pointers without changing the order of magnitude of the mass of the clock. A new element will enter the calculation, however, by the need for constructing wave packets which will remain, for a period $T$, with a high probability $1-\delta$, within a region of length $\lambda$. The characteristics of such a wave packet cannot be obtained from the uncertainty principles any more. However, a well known solution of the Schrödinger equation for a free particle,

$$\varphi = \frac{(\alpha/\pi)^{\frac{1}{4}}}{(\alpha + i\hbar t/M)^{\frac{1}{2}}} \exp\left(\frac{-\frac{1}{2}x^2}{\alpha + i\hbar t/M}\right), \qquad (12)$$

shows that the particle will remain with a probability $1-\delta$ in the interval of width $\lambda$ if

$$\frac{\frac{1}{4}\lambda^2\alpha}{\alpha^2 + \hbar^2T^2/M^2} > -\ln\delta. \qquad (13)$$

The maximum of the expression on the left is assumed for $\alpha = \hbar T/M$ which is then the best choice for the wave packet in question. It gives for $M$ the condition

$$M > 8\hbar T(-\ln\delta)/\lambda^2. \qquad (13a)$$

Except for the factor $(-\ln\delta)$, this is equivalent to (5). However, the logarithmic dependence of $M$ on $\delta$ is just the essential feature. It means that confining the wave packet in such a way that its average spread is

$$\Delta x = \lambda(-\ln\delta)^{-\frac{1}{2}} \qquad (13b)$$

can insure that the particle is within an interval $\lambda$ with a probability $1-\delta$. The momentum spread of the wave packet (12) is

$$\Delta p = \hbar/\alpha^{\frac{1}{2}} = (\hbar M/T)^{\frac{1}{2}} = (2\hbar/\lambda)(-2\ln\delta)^{\frac{1}{2}}, \qquad (13c)$$

if the mass is chosen close to its lower limit (13a). This is not the minimum of the momentum uncertainty;

it would be possible to reduce it by choosing a higher mass. However, $\Delta p$ will be used only for calculating the energy uncertainty and the resulting expression will already be at least of the same order of magnitude as the uncertainty in the energy of the light quanta which transmit the signal.

The preceding expressions apply for each of the particles which act as pointers. In order to obtain their wave functions, the packet (12) has to be given a suitable velocity $v$. This can be done by substituting $x-vt$ for $x$ in (12) and multiplying this with

$$\exp[iMvx/\hbar-\tfrac{1}{2}iMv^2t/\hbar]. \tag{14}$$

This has no effect on either $\Delta x$ or $\Delta p$. The reflection on the particles 1 and 3 which constitute the frame of the clock can be taken into account by the method of images, that is, by adding to the wave packet obtained by the multiplication of (12) by (14) similar wave packets, but with opposite sign, obtained by reflection on the $x=\pm l$ lines, and the images of these images, and so on. These operations will also leave the preceding equations essentially unchanged. If pointer $j$ has a period of $n_j\tau$ (hence $n_k=2n$), its average velocity will be

$$v_j=2l/n_j\tau, \tag{15}$$

and the accuracy of its reading must be

$$\tau_j=\tfrac{1}{2}n_{j-1}\tau;\quad \tau_1=\tau. \tag{15a}$$

In order to assure this accuracy with a probability of the order $\frac{1}{2}$, it would suffice to confine the wave packet of its pointer to

$$\lambda_j=v_j\tau_j=ln_{j-1}/n_j;\quad \lambda_1=2l/n_1. \tag{15b}$$

However, the reading of pointer $j$ must be correct with a probability $1-\delta_j$, where

$$\delta_j=1/2n_{j-1};\quad \delta_1=1/2. \tag{16}$$

It follows that the mass of pointer $j$ is at least

$$M_j=\frac{\hbar T}{l^2}\frac{n_j^2}{n_{j-1}^2}\ln(2n_{j-1}). \tag{16a}$$

Similarly,

$$\Delta x_j=(ln_{j-1}/n_j)(\ln 2n_{j-1})^{-\frac{1}{2}} \tag{16b}$$

and

$$\Delta p_j=(2\hbar n_j/ln_{j-1})(2\ln 2n_{j-1})^{\frac{1}{2}}. \tag{16c}$$

The last expression gives for the energy uncertainty of the pointer $j$

$$\epsilon_{cj}=v_j\Delta p_j=(4\hbar/\tau n_{j-1})(2\ln 2n_{j-1})^{\frac{1}{2}}, \tag{17a}$$

$$\epsilon_{c1}=4\hbar/\tau, \tag{17b}$$

while the energy uncertainty of the light signal which transmits the reading of this pointer is

$$\epsilon_{sj}=\hbar c/\Delta x_j=(\hbar cn_j/ln_{j-1})(\ln 2n_{j-1})^{\frac{1}{2}}. \tag{17}$$

For $l=c\tau$, this is about $n_j$ times greater than $\epsilon_{cj}$ but not too different from $\epsilon_{c1}$. This shows that if the clock is confined to a region in space which corresponds to its accuracy, the reading requirement does not increase its energy uncertainty substantially. The mass (16a) of the pointers is, at any rate, greater than the mass of the quanta needed to read them.

The choice of the $n_1, n_2, \cdots, n_k$ remains to be made. The simplest choice is to set $n_1=2$, $n_2=4$, $\cdots$, $n_j=2^{j+1}$, and $k=\log_2 n$. With this,

$$M=\sum M_j=\frac{8\hbar T}{l^2}4(\ln 2+\ln 4+\cdots+\ln 2n)$$
$$=\frac{32\hbar T}{l^2}\ln 2(1+2+\cdots+k+1)\approx\frac{\hbar T}{l^2}(\ln n)^2. \tag{18}$$

The last expression is correct only apart from numerical factors which appear to be considerably in excess of 1. The uncertainty of energy, for the same clock, becomes

$$\epsilon_c=\epsilon_{c1}+\epsilon_{c2}+\cdots+\epsilon_{ck}\approx 4\hbar/\tau, \tag{18a}$$

while

$$\epsilon_s=\epsilon_{s1}+\epsilon_{s2}+\cdots+\epsilon_{sk}\approx(\hbar c/l)(\ln n)^{\frac{3}{2}}. \tag{18b}$$

These expressions are already reasonably close to the lower limits (6) and (2a) so that a further substantial reduction seems impossible. At the same time, the idealizations used in the construction of the clock are not such as to raise too serious doubts concerning its realizability.