

# The Connection Between Spin and Statistics<sup>1</sup>

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In the following paper we conclude for the relativistically invariant wave equation for free particles: From postulate (I), according to which the energy must be positive, the necessity of *Fermi-Dirac* statistics for particles with arbitrary half-integral spin; from postulate (II), according to which observables on different space-time points with a space-like distance are commutable, the necessity of *Einstein-Bose* statistics for particles with arbitrary integral spin. It has been found useful to divide the quantities which are irreducible against Lorentz transformations into four symmetry classes which have a commutable multiplication like  $+1$ ,  $-1$ ,  $+\epsilon$ ,  $-\epsilon$  with  $\epsilon^2=1$ .

## §1. UNITS AND NOTATIONS

SINCE the requirements of the relativity theory and the quantum theory are fundamental for every theory, it is natural to use as units the vacuum velocity of light  $c$ , and Planck's constant divided by  $2\pi$  which we shall simply denote by  $\hbar$ . This convention means that all quantities are brought to the dimension of the power of a length by multiplication with powers of  $\hbar$  and  $c$ . The reciprocal length corresponding to the rest mass  $m$  is denoted by  $\kappa=mc/\hbar$ .

As time coordinate we use accordingly the length of the light path. In specific cases, however, we do not wish to give up the use of the imaginary time coordinate. Accordingly, a tensor index denoted by small Latin letters  $i$ , refers to the imaginary time coordinate and runs from 1 to 4. A special convention for denoting the complex conjugate seems desirable. Whereas for quantities with the index 0 an asterisk signifies the complex-conjugate in the ordinary sense (e.g., for the current vector  $S_i$  the quantity  $S_0^*$  is the complex conjugate of the charge density  $S_0$ ), in general  $U^*_{ik\dots}$  signifies: the complex-conjugate of  $U_{ik\dots}$  multiplied with  $(-1)^n$ , where  $n$  is the number of occurrences of

the digit 4 among the  $i, k, \dots$  (e.g.  $S_4=iS_0$ ,  $S_4^*=iS_0^*$ ).

Dirac's spinors  $u_\rho$  with  $\rho=1, \dots, 4$  have always a Greek index running from 1 to 4, and  $u_\rho^*$  means the complex-conjugate of  $u_\rho$  in the ordinary sense.

Wave functions, insofar as they are ordinary vectors or tensors, are denoted in general with capital letters,  $U_i, U_{ik}\dots$ . The symmetry character of these tensors must in general be added explicitly. As classical fields the electromagnetic and the gravitational fields, as well as fields with rest mass zero, take a special place, and are therefore denoted with the usual letters  $\varphi_i, f_{ik}=-f_{ki}$ , and  $g_{ik}=g_{ki}$ , respectively.

The energy-momentum tensor  $T_{ik}$  is so defined, that the energy-density  $W$  and the momentum density  $G_k$  are given in natural units by  $W=-T_{44}$  and  $G_k=-iT_{k4}$  with  $k=1, 2, 3$ .

## §2. IRREDUCIBLE TENSORS. DEFINITION OF SPINS

We shall use only a few general properties of those quantities which transform according to irreducible representations of the Lorentz group.<sup>2</sup> The proper Lorentz group is that continuous linear group the transformations of which leave the form

$$\sum_{k=1}^4 x_k^2 = \mathbf{x}^2 - x_0^2$$

invariant and in addition to that satisfy the condition that they have the determinant  $+1$

<sup>1</sup> This paper is part of a report which was prepared by the author for the Solvay Congress 1939 and in which slight improvements have since been made. In view of the unfavorable times, the Congress did not take place, and the publication of the reports has been postponed for an indefinite length of time. The relation between the present discussion of the connection between spin and statistics, and the somewhat less general one of Belinfante, based on the concept of charge invariance, has been cleared up by W. Pauli and F. J. Belinfante, *Physica* 7, 177 (1940).

<sup>2</sup> See B. L. v. d. Waerden, *Die gruppentheoretische Methode in der Quantentheorie* (Berlin, 1932).

and do not reverse the time. A tensor or spinor which transforms irreducibly under this group can be characterized by two integral positive numbers  $(p, q)$ . (The corresponding "angular momentum quantum numbers"  $(j, k)$  are then given by  $p=2j+1, q=2k+1$ , with integral or half-integral  $j$  and  $k$ .)<sup>\*</sup> The quantity  $U(j, k)$  characterized by  $(j, k)$  has  $p \cdot q = (2j+1)(2k+1)$  independent components. Hence to  $(0, 0)$  corresponds the scalar, to  $(\frac{1}{2}, \frac{1}{2})$  the vector, to  $(1, 0)$  the self-dual skew-symmetrical tensor, to  $(1, 1)$  the symmetrical tensor with vanishing spur, etc. Dirac's spinor  $u_p$  reduces to two irreducible quantities  $(\frac{1}{2}, 0)$  and  $(0, \frac{1}{2})$  each of which consists of two components. If  $U(j, k)$  transforms according to the representation

$$U_r' = \sum_{s=1}^{(2j+1)(2k+1)} \Lambda_{rs} U_s,$$

then  $U^*(k, j)$  transforms according to the complex-conjugate representation  $\Lambda^*$ . Thus for  $k=j, \Lambda^*=\Lambda$ . This is true only if the components of  $U(j, k)$  and  $U(k, j)$  are suitably ordered. For an arbitrary choice of the components, a similarity transformation of  $\Lambda$  and  $\Lambda^*$  would have to be added. In view of §1 we represent generally with  $U^*$  the quantity the transformation of which is equivalent to  $\Lambda^*$  if the transformation of  $U$  is equivalent to  $\Lambda$ .

The most important operation is the reduction of the product of two quantities

$$U_1(j_1, k_1) \cdot U_2(j_2, k_2)$$

which, according to the well-known rule of the composition of angular momenta, decompose into several  $U(j, k)$  where, independently of each other  $j, k$  run through the values

$$\begin{aligned} j &= j_1 + j_2, j_1 + j_2 - 1, \dots, |j_1 - j_2| \\ k &= k_1 + k_2, k_1 + k_2 - 1, \dots, |k_1 - k_2|. \end{aligned}$$

By limiting the transformations to the subgroup of space rotations alone, the distinction between the two numbers  $j$  and  $k$  disappears and  $U(j, k)$  behaves under this group just like the product of two irreducible quantities  $U(j)U(k)$  which in turn reduces into several irreducible

$U(l)$  each having  $2l+1$  components, with

$$l = j+k, j+k-1, \dots, |j-k|.$$

Under the space rotations the  $U(l)$  with integral  $l$  transform according to single-valued representation, whereas those with half-integral  $l$  transform according to double-valued representations. Thus the unreduced quantities  $T(j, k)$  with integral (half-integral)  $j+k$  are single-valued (double-valued).

If we now want to determine the spin value of the particles which belong to a given field it seems at first that these are given by  $l=j+k$ . Such a definition would, however, not correspond to the physical facts, for there then exists no relation of the spin value with the number of independent plane waves, which are possible in the absence of interaction) for given values of the components  $k_i$  in the phase factor  $\exp i(\mathbf{k}\mathbf{x})$ . In order to define the spin in an appropriate fashion,<sup>3</sup> we want to consider first the case in which the rest mass  $m$  of all the particles is different from zero. In this case we make a transformation to the rest system of the particle, where all the space components of  $k_i$  are zero, and the wave function depends only on the time. In this system we reduce the field components, which according to the field equations do not necessarily vanish, into parts irreducible against space rotations. To each such part, with  $r=2s+1$  components, belong  $r$  different eigenfunctions which under space rotations transform among themselves and which belong to a particle with spin  $s$ . If the field equations describe particles with only one spin value there then exists in the rest system only one such irreducible group of components. From the Lorentz invariance, it follows, for an arbitrary system of reference, that  $r$  or  $\sum r$  eigenfunctions always belong to a given arbitrary  $k_i$ . The number of quantities  $U(j, k)$  which enter the theory is, however, in a general coordinate system more complicated, since these quantities together with the vector  $k_i$  have to satisfy several conditions.

In the case of zero rest mass there is a special degeneracy because, as has been shown by Fierz, this case permits a gauge transformation of the

<sup>\*</sup> In the spinor calculus this is a spinor with  $2j$  undotted and  $2k$  dotted indices.

<sup>3</sup> See M. Fierz, *Helv. Phys. Acta* 12, 3 (1939); also L. de Broglie, *Comptes rendus* 208, 1697 (1939); 209, 265 (1939).

second kind.\* If the field now describes only one kind of particle with the rest mass zero and a certain spin value, then there are for a given value of  $k_i$  only two states, which cannot be transformed into each other by a gauge transformation. The definition of spin may, in this case, not be determined so far as the physical point of view is concerned because the total angular momentum of the field cannot be divided up into orbital and spin angular momentum by measurements. But it is possible to use the following property for a definition of the spin. If we consider, in the  $q$  number theory, states where only one particle is present, then not all the eigenvalues  $j(j+1)$  of the square of the angular momentum are possible. But  $j$  begins with a certain minimum value  $s$  and takes then the values  $s, s+1, \dots$ .<sup>4</sup> This is only the case for  $m=0$ . For photons,  $s=1; j=0$  is not possible for one single photon.<sup>5</sup> For gravitational quanta  $s=2$  and the values  $j=0$  and  $j=1$  do not occur.

In an arbitrary system of reference and for arbitrary rest masses, the quantities  $U$  all of which transform according to double-valued (single-valued) representations with half-integral (integral)  $j+k$  describe only particles with half-integral (integral) spin. A special investigation is required only when it is necessary to decide whether the theory describes particles with one single spin value or with several spin values.

### §3. PROOF OF THE INDEFINITE CHARACTER OF THE CHARGE IN CASE OF INTEGRAL AND OF THE ENERGY IN CASE OF HALF-INTEGRAL SPIN

We consider first a theory which contains only  $U$  with integral  $j+k$ , i.e., which describes particles with integral spins only. It is not assumed that only particles with one single spin value will be described, but all particles shall have integral spin.

\* By "gauge-transformation of the first kind" we understand a transformation  $U \rightarrow Ue^{i\alpha}$   $U^* \rightarrow U^*e^{-i\alpha}$  with an arbitrary space and time function  $\alpha$ . By "gauge-transformation of the second kind" we understand a transformation of the type

$$\varphi_k \rightarrow \varphi_k - \frac{1}{\epsilon} \frac{\partial \alpha}{\partial x_k}$$

as for those of the electromagnetic potentials.

<sup>4</sup> The general proof for this has been given by M. Fierz, *Helv. Phys. Acta* **13**, 45 (1940).

<sup>5</sup> See for instance W. Pauli in the article "Wellenmechanik" in the *Handbuch der Physik*, Vol. 24/2, p. 260.

We divide the quantities  $U$  into two classes (1) the "+1 class" with  $j$  integral,  $k$  integral (2) the "-1 class" with  $j$  half-integral,  $k$  half-integral.

The notation is justified because, according to the indicated rules about the reduction of a product into the irreducible constituents under the Lorentz group, the product of two quantities of the +1 class or two quantities of the -1 class contains only quantities of the +1 class, whereas the product of a quantity of the +1 class with a quantity of the -1 class contains only quantities of the -1 class. It is important that the complex conjugate  $U^*$  for which  $j$  and  $k$  are interchanged belong to the same class as  $U$ . As can be seen easily from the multiplication rule, tensors with even (odd) number of indices reduce only to quantities of the +1 class (-1 class). The propagation vector  $k_i$  we consider as belonging to the -1 class, since it behaves after multiplication with other quantities like a quantity of the -1 class.

We consider now a homogeneous and linear equation in the quantities  $U$  which, however, does not necessarily have to be of the first order. Assuming a plane wave, we may put  $k_i$  for  $-i\partial/\partial x_i$ . Solely on account of the invariance against the *proper* Lorentz group it must be of the typical form

$$\sum k U^+ = \sum U^-, \quad \sum k U^- = \sum U^+. \quad (1)$$

This typical form shall mean that there may be as many different terms of the same type present, as there are quantities  $U^+$  and  $U^-$ . Furthermore, among the  $U^+$  may occur the  $U^+$  as well as the  $(U^+)^*$ , whereas other  $U$  may satisfy reality conditions  $U = U^*$ . Finally we have omitted an *even* number of  $k$  factors. These may be present in arbitrary number in the term of the sum on the left- or right-hand side of these equations. It is now evident that these equations remain invariant under the substitution

$$k_i \rightarrow -k_i; \quad U^+ \rightarrow U^+, \quad [(U^+)^* \rightarrow (U^+)^*]; \\ U^- \rightarrow -U^-, \quad [(U^-)^* \rightarrow -(U^-)^*]. \quad (2)$$

Let us consider now tensors  $T$  of even rank (scalars, skew-symmetrical or symmetrical tensors of the 2nd rank, etc.), which are composed quadratically or bilinearly of the  $U$ 's. They are then composed solely of quantities with even  $j$

and even  $k$  and thus are of the typical form

$$T \sim \sum U^+ U^+ + \sum U^- U^- + \sum U^+ k U^-, \quad (3)$$

where again a possible even number of  $k$  factors is omitted and no distinction between  $U$  and  $U^*$  is made. Under the substitution (2) they remain unchanged,  $T \rightarrow T$ .

The situation is different for tensors of odd rank  $S$  (vectors, etc.) which consist of quantities with half-integral  $j$  and half-integral  $k$ . These are of the typical form

$$S \sim \sum U^+ k U^+ + \sum U^- k U^- + \sum U^- \quad (4)$$

and hence change the sign under the substitution (2),  $S \rightarrow -S$ . Particularly is this the case for the current vector  $s_i$ . To the transformation  $k_i \rightarrow -k_i$ , belongs for arbitrary wave packets the transformation  $x_i \rightarrow -x_i$  and it is remarkable that from the invariance of Eq. (1) against the proper Lorentz group alone there follows an invariance property for the change of sign of all the coordinates. In particular, the indefinite character of the current density and the total charge for even spin follows, since to every solution of the field equations belongs another solution for which the components of  $s_k$  change their sign. The definition of a definite particle density for even spin which transforms like the 4-component of a vector is therefore impossible.

We now proceed to a discussion of the somewhat less simple case of half-integral spins. Here we divide the quantities  $U$ , which have half-integral  $j+k$ , in the following fashion: (3) the “ $+\epsilon$  class” with  $j$  integral  $k$  half-integral, (4) the “ $-\epsilon$  class” with  $j$  half-integral  $k$  integral.

The multiplication of the classes (1),  $\dots$ , (4), follows from the rule  $\epsilon^2=1$  and the commutability of the multiplication. This law remains unchanged if  $\epsilon$  is replaced by  $-\epsilon$ .

We can summarize the multiplication law between the different classes in the following multiplication table:

	1	-1	$\epsilon$	$-\epsilon$
1	1	-1	$\epsilon$	$-\epsilon$
-1	-1	+1	$-\epsilon$	$+\epsilon$
$\epsilon$	$\epsilon$	$-\epsilon$	+1	-1
$-\epsilon$	$-\epsilon$	$\epsilon$	-1	+1

We notice that these classes have the multiplication law of Klein's “four-group.”

It is important that here the complex-conjugate quantities for which  $j$  and  $k$  are interchanged do not belong to the same class, so that

$$\begin{aligned} U^{+\epsilon}, (U^{-\epsilon})^* &\text{ belong to the } +\epsilon \text{ class} \\ U^{-\epsilon}, (U^{+\epsilon})^* &\text{ } -\epsilon \text{ class.} \end{aligned}$$

We shall therefore cite the complex-conjugate quantities explicitly. (One could even choose the  $U^{+\epsilon}$  suitably so that all quantities of the  $-\epsilon$  class are of the form  $(U^{+\epsilon})^*$ .)

Instead of (1) we obtain now as typical form

$$\begin{aligned} \sum k U^{+\epsilon} + \sum k (U^{-\epsilon})^* &= \sum U^{-\epsilon} + \sum (U^{+\epsilon})^* \\ \sum k U^{-\epsilon} + \sum k (U^{+\epsilon})^* &= \sum U^{+\epsilon} + \sum (U^{-\epsilon})^*, \end{aligned} \quad (5)$$

since a factor  $k$  or  $-i\partial/\partial x$  always changes the expression from one of the classes  $+\epsilon$  or  $-\epsilon$  into the other. As above, an even number of  $k$  factors have been omitted.

Now we consider instead of (2) the substitution

$$\begin{aligned} k_i \rightarrow -k_i; \quad U^{+\epsilon} \rightarrow i U^{+\epsilon}; \quad (U^{-\epsilon})^* \rightarrow i (U^{-\epsilon})^*; \\ (U^{+\epsilon})^* \rightarrow -i (U^{+\epsilon})^*; \quad U^{-\epsilon} \rightarrow -i U^{-\epsilon}. \end{aligned} \quad (6)$$

This is in accord with the algebraic requirement of the passing over to the complex conjugate, as well as with the requirement that quantities of the same class as  $U^{+\epsilon}, (U^{-\epsilon})^*$  transform in the same way. Furthermore, it does not interfere with possible reality conditions of the type  $U^{+\epsilon} = (U^{-\epsilon})^*$  or  $U^{-\epsilon} = (U^{+\epsilon})^*$ . Equations (5) remain unchanged under the substitution (6).

We consider again tensors of even rank (scalars, tensors of 2nd rank, etc.), which are composed bilinearly or quadratically of the  $U$  and their complex-conjugate. For reasons similar to the above they must be of the form

$$\begin{aligned} T \sim \sum U^{+\epsilon} U^{+\epsilon} + \sum U^{-\epsilon} U^{-\epsilon} + \sum U^{+\epsilon} k U^{-\epsilon} + \sum U^{+\epsilon} (U^{-\epsilon})^* + \sum U^{-\epsilon} (U^{+\epsilon})^* + \sum (U^{-\epsilon})^* k U^{-\epsilon} \\ + \sum (U^{+\epsilon})^* k U^{+\epsilon} + \sum (U^{-\epsilon})^* k (U^{+\epsilon})^* + \sum (U^{-\epsilon})^* (U^{-\epsilon})^* + \sum (U^{+\epsilon})^* (U^{+\epsilon})^*. \end{aligned} \quad (7)$$

Furthermore, the tensors of odd rank (vectors, etc.) must be of the form

$$\begin{aligned} S \sim \sum U^{+\epsilon} k U^{+\epsilon} + \sum U^{-\epsilon} k U^{-\epsilon} + \sum U^{+\epsilon} U^{-\epsilon} + \sum U^{+\epsilon} k (U^{-\epsilon})^* + \sum U^{-\epsilon} k (U^{+\epsilon})^* + \sum U^{-\epsilon} (U^{-\epsilon})^* \\ + \sum U^{+\epsilon} (U^{+\epsilon})^* + \sum (U^{-\epsilon})^* k (U^{-\epsilon})^* + \sum (U^{+\epsilon})^* k (U^{+\epsilon})^* + \sum (U^{-\epsilon})^* (U^{+\epsilon})^*. \end{aligned} \quad (8)$$

The result of the substitution (6) is now the opposite of the result of the substitution (2): the tensors of even rank change their sign, the tensors of odd rank remain unchanged:

$$T \rightarrow -T; \quad S \rightarrow +S. \quad (9)$$

In case of half-integral spin, therefore, a positive definite energy density, as well as a positive definite total energy, is impossible. The latter follows from the fact, that, under the above substitution, the energy density in every space-time point changes its sign as a result of which the total energy changes also its sign.

It may be emphasized that it was not only unnecessary to assume that the wave equation is of the first order,\* but also that the question is left open whether the theory is also invariant with respect to space reflections ( $\mathbf{x}' = -\mathbf{x}, x_0' = x_0$ ). This scheme covers therefore also Dirac's two component wave equations (with rest mass zero).

These considerations do not prove that for integral spins there always exists a definite energy density and for half-integral spins a definite charge density. In fact, it has been shown by Fierz<sup>6</sup> that this is not the case for spin  $> 1$  for the densities. There exists, however (in the  $c$  number theory), a definite total charge for half-integral spins and a definite total energy for the integral spins. The spin value  $\frac{1}{2}$  is discriminated through the possibility of a definite charge density, and the spin values 0 and 1 are discriminated through the possibility of defining a definite energy density. Nevertheless, the present theory permits arbitrary values of the spin quantum numbers of elementary particles as well as arbitrary values of the rest mass, the electric charge, and the magnetic moments of the particles.

#### §4. QUANTIZATION OF THE FIELDS IN THE ABSENCE OF INTERACTIONS. CONNECTION BETWEEN SPIN AND STATISTICS

The impossibility of defining in a physically satisfactory way the particle density in the case of integral spin and the energy density in the case of half-integral spins in the  $c$ -number theory

\* But we exclude operations like  $(k^2 + \kappa^2)^{\frac{1}{2}}$ , which operate at finite distances in the coordinate space.

<sup>6</sup> M. Fierz, *Helv. Phys. Acta* 12, 3 (1939).

is an indication that a satisfactory interpretation of the theory within the limits of the one-body problem is not possible.\* In fact, all relativistically invariant theories lead to particles, which in external fields can be emitted and absorbed in pairs of opposite charge for electrical particles and singly for neutral particles. The fields must, therefore, undergo a second quantization. For this we do not wish to apply here the canonical formalism, in which time is unnecessarily sharply distinguished from space, and which is only suitable if there are no supplementary conditions between the canonical variables.<sup>7</sup> Instead, we shall apply here a generalization of this method which was applied for the first time by Jordan and Pauli to the electromagnetic field.<sup>8</sup> This method is especially convenient in the absence of interaction, where all fields  $U^{(r)}$  satisfy the wave equation of the second order

$$\square U^{(r)} - \kappa^2 U^{(r)} = 0,$$

where

$$\square \equiv \sum_{k=1}^4 \frac{\partial^2}{\partial x_k^2} = \Delta - \frac{\partial^2}{\partial x_0^2}$$

and  $\kappa$  is the rest mass of the particles in units  $\hbar/c$ .

An important tool for the second quantization is the invariant  $D$  function, which satisfies the wave equation (9) and is given in a periodicity volume  $V$  of the eigenfunctions by

$$D(\mathbf{x}, x_0) = \frac{1}{V} \sum \exp [i(\mathbf{k}\mathbf{x})] \frac{\sin k_0 x_0}{k_0} \quad (10)$$

or in the limit  $V \rightarrow \infty$

$$D(\mathbf{x}, x_0) = \frac{1}{(2\pi)^3} \int d^3k \exp [i(\mathbf{k}\mathbf{x})] \frac{\sin k_0 x_0}{k_0}. \quad (11)$$

\* The author therefore considers as not conclusive the original argument of Dirac, according to which the field equation must be of the first order.

<sup>7</sup> On account of the existence of such conditions the canonical formalism is not applicable for spin  $> 1$  and therefore the discussion about the connection between spin and statistics by J. S. de Wet, *Phys. Rev.* 57, 646 (1940), which is based on that formalism is not general enough.

<sup>8</sup> The consistent development of this method leads to the "many-time formalism" of Dirac, which has been given by P. A. M. Dirac, *Quantum Mechanics* (Oxford, second edition, 1935).

By  $k_0$  we understand the positive root

$$k_0 = +(k^2 + \kappa^2)^{1/2}. \quad (12)$$

The  $D$  function is uniquely determined by the conditions:

$$\square D - \kappa^2 D = 0; \quad D(\mathbf{x}, 0) = 0; \quad \left(\frac{\partial D}{\partial x_0}\right)_{x_0=0} = \delta(\mathbf{x}). \quad (13)$$

For  $\kappa = 0$  we have simply

$$D(\mathbf{x}, x_0) = \{\delta(r - x_0) - \delta(r + x_0)\} / 4\pi r. \quad (14)$$

This expression also determines the singularity of  $D(\mathbf{x}, x_0)$  on the light cone for  $\kappa \neq 0$ . But in the latter case  $D$  is no longer different from zero in the inner part of the cone. One finds for this region<sup>9</sup>

$$D(\mathbf{x}, x_0) = -\frac{1}{4\pi r} \frac{\partial}{\partial r} F(r, x_0)$$

with

$$F(r, x_0) = \begin{cases} J_0[\kappa(x_0^2 - r^2)^{1/2}] & \text{for } x_0 > r \\ 0 & \text{for } r > x_0 > -r \\ -J_0[\kappa(x_0^2 - r^2)^{1/2}] & \text{for } -r > x_0. \end{cases} \quad (15)$$

The jump from  $+$  to  $-$  of the function  $F$  on the light cone corresponds to the  $\delta$  singularity of  $D$  on this cone. For the following it will be of decisive importance that  $D$  vanish in the exterior of the cone (i.e., for  $r > x_0 > -r$ ).

The form of the factor  $d^3k/k_0$  is determined by the fact that  $d^3k/k_0$  is invariant on the hyperboloid ( $k$ ) of the four-dimensional momentum space ( $\mathbf{k}, k_0$ ). It is for this reason that, apart from  $D$ , there exists just one more function which is invariant and which satisfies the wave equation (9), namely,

$$D_1(\mathbf{x}, x_0) = \frac{1}{(2\pi)^3} \int d^3k \exp [i(\mathbf{k}\mathbf{x})] \frac{\cos k_0 x_0}{k_0}. \quad (16)$$

For  $\kappa = 0$  one finds

$$D_1(\mathbf{x}, x_0) = \frac{1}{2\pi^2} \frac{1}{r^2 - x_0^2}. \quad (17)$$

In general it follows

$$D_1(\mathbf{x}, x_0) = \frac{1}{4\pi} \frac{1}{r} \frac{\partial}{\partial r} F_1(r, x_0)$$

$$F_1(r, x_0) = \begin{cases} N_0[\kappa(x_0^2 - r^2)^{1/2}] & \text{for } x_0 > r \\ -iH_0^{(1)}[i\kappa(r^2 - x_0^2)^{1/2}] & \text{for } r > x_0 > -r \\ N_0[\kappa(x_0^2 - r^2)^{1/2}] & \text{for } -r > x_0. \end{cases} \quad (18)$$

Here  $N_0$  stands for Neumann's function and  $H_0^{(1)}$  for the first Hankel cylinder function. The strongest singularity of  $D$ , on the surface of the light cone is in general determined by (17).

We shall, however, expressively postulate in the following that all physical quantities at finite distances exterior to the light cone (for  $|x_0' - x_0''| < |\mathbf{x}' - \mathbf{x}''|$ ) are commutable.\* It follows from this that the bracket expressions of all quantities which satisfy the force-free wave equation (9) can be expressed by the function  $D$  and (a finite number) of derivatives of it without using the function  $D_1$ . This is also true for brackets with the  $+$  sign, since otherwise it would follow that gauge invariant quantities, which are constructed bilinearly from the  $U^{(r)}$ , as for example the charge density, are noncommutable in two points with a space-like distance.<sup>10</sup>

The justification for our postulate lies in the fact that measurements at two space points with a space-like distance can never disturb each other, since no signals can be transmitted with velocities greater than that of light. Theories which would make use of the  $D_1$  function in their quantization would be very much different from the known theories in their consequences.

At once we are able to draw further conclusions about the number of derivatives of  $D$  function which can occur in the bracket expressions, if we take into account the invariance of the theories under the transformations of the restricted Lorentz group and if we use the results of the preceding section on the class division of the tensors. We assume the quantities  $U^{(r)}$  to be ordered in such a way that each field component is composed only of quantities of the same class.

\* For the canonical quantization formalism this postulate is satisfied implicitly. But this postulate is much more general than the canonical formalism.

<sup>10</sup> See W. Pauli, Ann. de l'Inst. H. Poincaré 6, 137 (1936), esp. §3.

<sup>9</sup> See P. A. M. Dirac, Proc. Camb. Phil. Soc. 30, 150 (1934).

We consider especially the bracket expression of a field component  $U^{(r)}$  with its own complex conjugate

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')].$$

We distinguish now the two cases of half-integral and integral spin. In the former case this expression transforms according to (8) under Lorentz transformations as a tensor of odd rank. In the second case, however, it transforms as a tensor of even rank. Hence we have for half-integral spin

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] = \text{odd number of derivatives of the function } D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0'') \quad (19a)$$

and similarly for integral spin

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] = \text{even number of derivatives of the function } D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0''). \quad (19b)$$

This must be understood in such a way that on the right-hand side there may occur a complicated sum of expressions of the type indicated. We consider now the following expression, which is symmetrical in the two points

$$X \equiv [U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] + [U^{(r)}(\mathbf{x}'', x_0''), U^{*(r)}(\mathbf{x}', x_0')]. \quad (20)$$

Since the  $D$  function is even in the space coordinates odd in the time coordinate, which can be seen at once from Eqs. (11) or (15), it follows from the symmetry of  $X$  that  $X = \text{even number of space-like times odd numbers of time-like derivatives of } D(\mathbf{x}' - \mathbf{x}'', x_0' - x_0'')$ . This is fully consistent with the postulate (19a) for half-integral spin, but in contradiction with (19b) for integral spin unless  $X$  vanishes. We have therefore the result for integral spin

$$[U^{(r)}(\mathbf{x}', x_0'), U^{*(r)}(\mathbf{x}'', x_0'')] + [U^{(r)}(\mathbf{x}'', x_0''), U^{*(r)}(\mathbf{x}', x_0')] = 0. \quad (21)$$

So far we have not distinguished between the two cases of Bose statistics and the exclusion principle. In the former case, one has the ordinary bracket with the  $-$  sign, in the latter case,

according to Jordan and Wigner, the bracket

$$[A, B]_+ = AB + BA$$

with the  $+$  sign. By inserting the brackets with the  $+$  sign into (20) we have an algebraic contradiction since the left-hand side is essentially positive for  $x' = x''$  and cannot vanish unless both  $U^{(r)}$  and  $U^{*(r)}$  vanish.\*

Hence we come to the result: *For integral spin the quantization according to the exclusion principle is not possible. For this result it is essential, that the use of the  $D_1$  function in place of the  $D$  function be, for general reasons, discarded.*

On the other hand, it is formally possible to quantize the theory for half-integral spins according to Einstein-Bose-statistics, but according to the general result of the preceding section the energy of the system would not be positive. Since for physical reasons it is necessary to postulate this, we must apply the exclusion principle in connection with Dirac's hole theory.

For the positive proof that a theory with a positive total energy is possible by quantization according to Bose-statistics (exclusion principle) for integral (half-integral) spins, we must refer to the already mentioned paper by Fierz. In another paper by Fierz and Pauli<sup>11</sup> the case of an external electromagnetic field and also the connection between the special case of spin 2 and the gravitational theory of Einstein has been discussed.

In conclusion we wish to state, that according to our opinion the connection between spin and statistics is one of the most important applications of the special relativity theory.

\* This contradiction may be seen also by resolving  $U^{(r)}$  into eigenvibrations according to

$$U^{(r)}(\mathbf{x}, x_0) = V^{-1} \sum_k \{ U_{+}^{*(r)}(k) \exp [i \{ -(k\mathbf{x}) + k_0 x_0 \}] + U_{-}^{(r)}(k) \exp [i \{ (k\mathbf{x}) - k_0 x_0 \}] \}$$

$$U^{(r)}(\mathbf{x}, x_0) = V^{-1} \sum_k \{ U_{+}^{(r)}(k) \exp [i \{ (k\mathbf{x}) - k_0 x_0 \}] + U_{-}^{*(r)}(k) \exp [i \{ -(k\mathbf{x}) + k_0 x_0 \}] \}.$$

The equation (21) leads then, among others, to the relation

$$[U_{+}^{*(r)}(k), U_{+}^{(r)}(k)] + [U_{-}^{(r)}(k), U_{-}^{*(r)}(k)] = 0,$$

a relation, which is not possible for brackets with the  $+$  sign unless  $U_{\pm}^{(r)}(k)$  and  $U_{\pm}^{*(r)}(k)$  vanish.

<sup>11</sup> M. Fierz and W. Pauli, Proc. Roy. Soc. A173, 211 (1939).