Lifetimes of Nuclear Isomers*

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Lifetimes of nuclear isomers for gamma-transitions are calculated theoretically on the basis of various independent particle models; e.g., single proton, single neutron, and states of several particles (Sec. II). The calculations of this paper are essentially restricted to the most common type of transition *viz.*, multipole order equal to spin change. The lifetime is expressed in terms of a matrix element, M, whose theoretical value depends on the particular model of the nucleus. Radial integrals are calculated numerically, assuming that the nuclear wave functions are given by single particle wave functions for a spherical square well.

Empirical values of M^2 can be deduced from measured isomeric lifetimes, corrected for internal conversion. An analysis of empirical M^2 for some gamma transitions points to a number of regularities which, in general, speak in favor of an independent particle model (Sec. III).

The regularities are the following:

Empirical values of M^2 for $M4$ transitions are of order unity and show little scattering and no distinction between odd proton and odd neutron nuclei. The lack of scattering within each group of transitions is consistent with predictions of a single particle model. However, according to this model, one would expect odd proton nuclei to have lifetimes about half as large as odd neutron nuclei for the same transition energy, and also would expect lifetimes about 1/10 as large as found empirically. Empirical values of M^2 for $M4$ transitions appear to be larger for transitions in nuclei with nearly closed shells.

According to an independent particle model, M^2 for $E3$ transitions of energy 100 kev should be of order 10^{-8} for single neutron transitions, and vanish for many particle transitions, such as those between $p_{\frac{1}{2}}$ and 7/2+states. The fact that empirical M^2 for $E3$ transitions are small can be interpreted as resulting from small deviations from an independent particle model. In fact, empirical M^2 for transitions between $p_{\frac{1}{2}}$ and 7/2+states in odd-neutron nuclei appear to be smaller the more nearly the nucleus can be represented as a closed shell nucleus.

The empirical value of M^2 for an $M1$ isomeric transition in Li^7 is slightly larger than expected according to an independent particle model.

A graph of energy levels for a spherical square well potential is presented (Appendix, Fig. 2).

I. INTRODUCTION

It was first suggested by v. Weizsacker,[1] and has become generally accepted, that nuclear isomeric states decay into each other by gamma-emission, but that the lifetime is large if the spin of the two states differs by several units of $\hbar$.

The emitted quantum may carry off angular momentum $L(\geq 1)$, giving rise to electric or magnetic 2^L pole radiation (denoted in this paper by EL or ML), according to whether the quantum has parity $(-1)^L$ or $(-1)^{L-1}$. It follows that the selection rules for EL and ML radiation, i.e., radiation of various multipolarities, are

$$|I_i - I_f| \leq L \leq I_i + I_f, \tag{1}$$

parity change $(-1)^L$ for EL, $(-1)^{L-1}$ for ML. Here I_i and I_f denote the spins of the initial and final states, respectively.

Electromagnetic radiation is not the only mode of decay for a nuclear excited state. Instead, an internal conversion electron may be emitted; i.e., an orbital electron may be ejected and carry off the energy of excitation. The number of electrons emitted from a given shell per quantum is called the conversion coefficient for that shell. Values of conversion coefficients depend on the multipolarity of the transition and on the electronic configuration. However, they do not depend on the detailed structure of the nucleus, and can therefore be calculated, in principle, to the same accuracy as spectroscopic problems.

K-conversion coefficients have been calculated by Rose *et al.*[2] neglecting the effect of screening, for energies above 150 kev, and by Reitz[3] for selected cases, including the effect of screening. Calculations of K, L_{I}, L_{II} and L_{III} conversion coefficients over a wide range of energies and including screening effects are now in progress.[4]

Conversion data are extremely useful for obtaining multipolarity of a transition, but not for obtaining information regarding detailed nuclear structure (apart from the spin and parity change). Goldhaber and Sunyar[5] have made multipolarity assignments for many transitions by comparing observed K-coefficients with Rose's theoretical values, and also from a semi-empirical analysis of K/L ratios, which indicates that K/L ratios are a function of Z^2/E and multipolarity alone (except for $M1$ transitions).[6] The work of Mihelich and Church[7] indicates that ratios of L-subshell conversion coefficients can be of use for assigning multipolarities and for analyzing transitions which involve a mixture of multipolarities.

The lifetime of an excited state for gamma-emission depends not only on the multipolarity and energy of the transition, but also on the detailed structure of the nucleus. It is, thus, not possible to identify the multi-

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[1] C. F. v. Weizsacker, Naturwiss. 24, 813 (1936).
[2] Rose, Goertzel, Spinrad, Harr, and Strong, Phys. Rev. 83, 79 (1951).
[3] J. R. Reitz, Phys. Rev. 77, 10 (1950).
[4] M. E. Rose and G. Goertzel (to be published).
[5] M. Goldhaber and A. W. Sunyar, Phys. Rev. 83, 906 (1951).
[6] M. Goldhaber (private communication).
[7] J. W. Mihelich and E. L. Church, Phys. Rev. 85, 733 (1952); J. W. Mihelich, Phys. Rev. 87, 646 (1952).

polarity of a transition from a lifetime measurement alone, without specific assumptions as to nuclear structure.

However, information regarding nuclear structure can be obtained by comparing the measured lifetime of a transition with the lifetime calculated using a specific nuclear model, e.g., the independent particle model,[8] provided the multipolarity of the transition has already been identified, say, from conversion data.

II. THEORETICAL CALCULATIONS OF LIFETIMES FOR GAMMA-DECAY

a. Single Proton for Spin Change Equal to Multiple Order-Central Potential

The transition probability per unit time for a system to undergo an electromagnetic transition from an initial state i to a final state f is given by the well-known equation[9]

$$W=\frac{2\pi}{\hbar}|\mathcal{H}_{fi}'|^2N(E). \tag{2}$$

Here $\mathcal{H}_{fi}'$ is the matrix element of the electromagnetic interaction between particles and field, $N(E)$ is the number of final states available per unit energy interval. The total nonrelativistic Hamiltonian of proton and electromagnetic field can be written[10]

$$\mathcal{H}=\frac{(\mathbf{p}-e\mathbf{A}/c)^2}{2M}+V(r)-\mu_P\frac{e\hbar}{2Mc}(\boldsymbol{\sigma}\cdot\mathbf{H})+\textstyle\sum_k n_k\hbar\omega_k. \tag{3}$$

Here M denotes mass of proton, μ_P is the proton magnetic moment in nuclear magnetrons. The vector potential, normalized to one quantum per unit volume is given by

$$\mathbf{A}=\boldsymbol{\varepsilon}(2\pi\hbar c^2/\omega)^{\frac{1}{2}}e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{4}$$

The radiation interaction is contained in (3) and is given by

$$\mathcal{H}'=-\frac{e}{Mc}\mathbf{p}\cdot\mathbf{A}-\mu_P\frac{e\hbar}{2Mc}(\boldsymbol{\sigma}\cdot\mathbf{H}). \tag{5}$$

[The term $(e^2/2Mc^2)\mathbf{A}^2$ is neglected here, as are second order perturbation terms, both of which give rise to double quantum emission, with a transition probability which is usually several orders of magnitude less than the transition probability for single quantum emission.][11,12] The first term in (5) is the interaction energy of a point charge with an electromagnetic field, while the second term is the interaction energy of an intrinsic magnetic moment with the field. The transition probability between states is calculated using (2), summing over polarizations, and integrating over all directions of emission:

$$W=\frac{2\omega}{\hbar c}\int_{4\pi}\frac{d\Omega_n}{4\pi}\sum_{\epsilon}|J_{fi}|^2. \tag{6}$$

Here $d\Omega_n$ is the solid angle for direction of emission,

$$J_{fi}=\left|\left(\frac{e}{Mc}(\mathbf{p}\cdot\boldsymbol{\varepsilon})+\frac{i\omega}{c}\mu_P\frac{e\hbar}{2Mc}(\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}')\right)e^{i\mathbf{k}\cdot\mathbf{r}}\right|_{fi}. \tag{7}$$

$\boldsymbol{\varepsilon}'=\mathbf{n}\times\boldsymbol{\varepsilon}$, and $\mathbf{n}=\mathbf{k}c/\omega$, the unit vector in direction of emission. The exponential in (7) can be expanded in ascending powers of $\mathbf{k}\cdot\mathbf{r}$, which is assumed to be $\ll 1$.

Using elementary relations of quantum mechanics, one can write J_{fi} as follows:

$$J_{fi}=\sum_{L=1}^{\infty}\frac{e}{L!}\left(\frac{i\omega}{c}\right)^L[(\mathbf{r}\cdot\boldsymbol{\varepsilon})(\mathbf{r}\cdot\mathbf{n})^{L-1}]_{fi}$$
$$+\sum_{L=1}^{\infty}\frac{e\hbar}{Mc}\frac{1}{(L-1)!}\left(\frac{i\omega}{c}\right)^L\left[\left(\frac{\mathbf{l}\cdot\boldsymbol{\varepsilon}'}{L+1}+\frac{\mu_P\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}'}{2}\right)\right.$$
$$\left.\times\frac{(\mathbf{r}\cdot\mathbf{n})^{L-1}}{2}+\frac{(\mathbf{r}\cdot\mathbf{n})^{L-1}}{2}\left(\frac{\mathbf{l}\cdot\boldsymbol{\varepsilon}'}{L+1}+\frac{\mu_P\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}'}{2}\right)\right]. \tag{8}$$

Here $\mathbf{l}$ is the orbital angular momentum operator in units of $\hbar$.

A transition between two given states can usually proceed by radiation of several multipolarities, consistent with the selection rules (1). In this paper, only those transitions are considered for which the radius of the nucleus, a, is much smaller than the wavelength of the radiation, $\omega a/c\ll 1$; e.g., most nuclear isomeric transitions. For such cases, the calculations presented here, and also independent calculations of Weisskopf[13] and of Stech,[14] indicate, in accord with experimental evidence, that the transition will go primarily by the lowest multipole order permitted by the selection rules, *viz.*, EL for $\Delta I=L$, parity change $(-1)^L$. The only exceptions to this found experimentally to date are a few $M1+E2$ mixtures.[5,7]

Interaction terms which give rise to pure 2^L pole radiation transform under space rotation of the nucleus as the components of the irreducible tensor of order L,[15] denoted by $\mathfrak{D}^{(L)}$. For transitions involving $\Delta I=L$, but only for these, terms which result from an expansion of the interaction into powers of $\mathbf{k}\cdot\mathbf{r}$ transform as $\mathfrak{D}^{(L)}$. In particular, the Lth term of the first and second series of (8) corresponds to an EL and ML transition, respectively.

To evaluate (8) and the transition probability (6), one uses the fact that for a central potential, the wave

[8] M. G. Mayer, Phys. Rev. **78**, 16, 22 (1950).
[9] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1949), first edition, p. 193.
[10] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, London, 1947), second edition, p. 91.
[11] M. Goeppert-Mayer, Ann. Physik **9**, 273 (1931).
[12] R. G. Sachs, Phys. Rev. **57**, 194 (1940).
[13] V. F. Weisskopf, Phys. Rev. **83**, 1073 (1951).
[14] B. Stech, Z. Naturforsch. **7a**, 401 (1952).
[15] E. Wigner, *Gruppentheorie und iher Anwendung auf die Quantenmechanik der Atom Spektren* (Edwards Brothers, Ann Arbor, Michigan, 1944), p. 164.

function of initial and final state of the proton can be written as product of a radial function R and of an angular function $\theta_{l,I}{}^{m}$ (for intrinsic spin $\frac{1}{2}$). Integrations over the angular functions can be performed exactly, since they involve only the symmetry properties of the nucleus. However, values of integrals over radial functions depend on the particular nuclear potential chosen.

Since the transition probability and thus the gamma-decay lifetime $\tau_\gamma = 1/W$ is independent of the direction quantum number of the initial state m_i,[16] one can calculate the gamma-ray lifetime for a transition for which $I_i = I_f + L$ and take $m_i = I_i$; then the summation over final m-substates reduces to a single term, with $m_f = I_f$. Thus the gamma-ray lifetime is given by the following equations[17]:

$$\tau_{\gamma EL} \underset{\substack{\Delta I = L \\ \Delta l = L}}{=} \left[\frac{2(L+1)}{L} \frac{1}{(2L+1!!)^2} \frac{e^2}{\hbar c}\omega\left(\frac{\omega a}{c}\right)^{2L} SM_{EL}{}^2\right]^{-1}, \quad (9a)$$

$$\tau_{\gamma ML} \underset{\substack{\Delta I = L \\ \Delta l = L-1}}{=} \left[\frac{2(L+1)}{L} \frac{1}{(2L+1!!)^2} \frac{e^2}{\hbar c}\omega\left(\frac{\omega a}{c}\right)^{2L} \times\left(\frac{\hbar}{Mca}\right)^2 SM_{ML}{}^2\right]^{-1}, \quad (9b)$$

where

$$M_{EL}{}^2 = |(r/a)^L|_{fi}{}^2, \quad (10a)$$

$$M_{ML}{}^2 = \left(\mu_P L - \frac{L}{L+1}\right)^2 \left|\left(\frac{r}{a}\right)^{L-1}\right|_{fi}^2, \quad (10b)$$

$$2L+1!! = 1\times3\times5\times\cdots\times(2L+1), \quad (11)$$

$$|(r/a)^L|_{fi} = \int_0^\infty R_f(r/a)^L R_i r^2 dr, \quad (12)$$

the radial integral in units of a^L, S is a statistical factor, whose value depends only on the spins involved in the transition and is defined as

$$S(I_i, L, I_f) = 4\pi \sum_{mf}\sum_{m}\left|\int_{4\pi} \theta_{l_f, I_f}{}^{m_f *} Y_L{}^{m} \theta_{l_i, I_i}{}^{m_i} d\Omega\right|^2. \quad (13)$$

$Y_L{}^m$ are spherical harmonics.

This gives

$$S(I_i, L, I_f) = \frac{(I_i - \frac{1}{2})! \times (2L+1)!! \times (2I_f)!!}{(2I_i)!! \times L! \times (I_f - \frac{1}{2})!} \quad (14a)$$

for $I_i = I_f + L$, namely, when the spin of the initial state is larger than the spin of the final state. For transitions with $I_f = I_i + L$ the statistical factor is given as follows, in accordance with a simple statistical weight argument:

$$S(I_i, L, I_f) = S(I_i + L, L, I_i) \times \frac{2I_i + 2L + 1}{2I_i + 1}. \quad (14b)$$

Note that S equals 1 for any transition for which $I_f = \frac{1}{2}$ and $I_i = L + \frac{1}{2}$. M is the only term in the equation which depends on the detailed nuclear structure. It is called here the "matrix element." According to Eqs. (10), M is expected to be of order unity, if the radial integral is (see Sec. IIg). To bring Eqs. (9) into correspondence with Weisskopf's equations,[13] one writes them as follows:

$$\tau_{\gamma EL} = \frac{L(2L+1!!)^2}{4.4(L+1)}\left(\frac{197\text{ Mev}}{\hbar\omega}\right)^{2L+1} \times (a \text{ in } 10^{-13}\text{ cm})^{-2L} 10^{-21} S^{-1} M_{EL}{}^{-2}\text{ sec}, \quad (15a)$$

$$\tau_{\gamma ML} = \frac{L(2L+1)!!}{0.19(L+1)}\left(\frac{197\text{ Mev}}{\hbar\omega}\right)^{2L+1} \times (a \text{ in } 10^{-13}\text{ cm})^{-(2L-2)} \cdot 10^{-21} S^{-1} M_{ML}{}^{-2}\text{ sec}. \quad (15b)$$

TABLE I. Values of S for $E1$ and $E2$ transitions.

I_f	S_{E1}	S_{E2}
I_i+2	$\cdots$	$\frac{15}{32}\frac{(2I_i+3)(2I_i+5)}{(I_i+1)(I_i+2)}$
I_i+1	$\frac{3}{4}\frac{(2I_i+3)}{(I_i+1)}$	$\frac{15}{16}\frac{(2I_i+3)}{I_i(I_i+1)(I_i+2)}$
I_i	$\frac{1}{4I_i(I_i+1)}$	$\frac{5}{16}\frac{(2I_i-1)(2I_i+3)}{(I_i-1)(I_i)}$
I_i-1	$\frac{3}{4}\frac{(2I_i-1)}{I_i}$	$\frac{15}{16}\frac{(2I_i-1)}{(I_i-1)(I_i)(I+1)}$
I_i-2	$\cdots$	$\frac{15}{32}\frac{(2I_i-3)(2I_i-1)}{(I_i-1)(I_i)}$

b. Electric Transitions for Spin Change Less than Multipole Order

The EL transition probability now has to be derived by expanding the radiation interaction explicitly into terms which transform as $\mathfrak{D}^{(L)}$ under space rotation of the nucleus. It can be shown, and is stated here without proof, that Eqs. (9a), (10a), (13) still hold in the limit $\omega a/c \ll 1$. For a given I_i, L, I_f, of the four possible kinds of transitions ($l_i = I_i \pm \frac{1}{2}$, $l_f = I_f \pm \frac{1}{2}$), two have the parity change $(-1)^L$ required for an EL transition, *viz.*, those transitions for which $l_i - l_f - L$ is an even integer. S is the same for these two transitions, as can be seen by inserting the relation

$$\frac{\boldsymbol{\sigma}\cdot\mathbf{r}}{r}\theta_{I\pm\frac{1}{2}, I}{}^{m} = \theta_{I\mp\frac{1}{2}, I}{}^{m} \quad (16)$$

into (13), and taking into account the fact that $Y_L{}^m$ commutes with $\boldsymbol{\sigma}$. Thus S for electric transition is a

[16] S. R. de Groot and H. A. Tolhoek, Physica 15, 833 (1949).
[17] S. A. Moszkowski, Phys. Rev. 83, 1071 (1951).

function of I_i, L and L_f alone. Table I shows values of S for $E1$ and $E2$ transitions.

c. Transitions of a Single Neutron

The calculation of transition probabilities for a single neutron is similar to the calculation for a single proton. The interaction Hamiltonian is given, as before, by (5). However, the first term vanishes due to the absence of electric charge of the neutron. More rigorously, if one considers the nucleus as a two-body system of neutron and core moving about a common center of gravity, one obtains an electric moment term due to the motion of the core.

For this model of the nucleus

$$\mathcal{H}'(\mathbf{r})=-\frac{Ze}{Mc}\left[\mathbf{p}\cdot\mathbf{A}\left(\frac{-r}{A}\right)\right]-\mu_N\frac{eh}{2Mc}\left[\boldsymbol{\sigma}\cdot\mathbf{H}(\mathbf{r})\right], \quad (17)$$

where μ_N is the neutron magnetic moment in nuclear magnetons.

Thus, transitions between single neutron states involving $\Delta I=L$ and parity change $(-1)^{L-1}$ will proceed by magnetic 2^L pole radiation as in the single proton case. With respect to Eqs. (9b), obtained for the single proton case, the only change is that the term $[\mu_P L-(L/L+1)]^2$ is replaced by $(\mu_N L)^2$. However, electric 2^L pole transitions between states involving $\Delta I=L$ and parity change $(-1)^L$ now have probabilities only $(Z/A^L)^2$ as large as for the single proton case. One can also get a contribution to the transition probability from the second term in (17), because of interaction of intrinsic magnetic moment with the electromagnetic field.[13] It is easy to convince oneself that the EL and $M(L+1)$ transition probabilities resulting from intrinsic magnetic moment are smaller than the corresponding EL probabilities for a single proton (9a) (resulting from the EL moment) by a factor of order $(\hbar\omega/Mc^2)^2$. It is thus seen that the EL matrix elements for gamma-ray transitions involving $L\geq 2$, for odd neutron nuclei, assuming an independent particle model, are expected to be much smaller than 1.

d. Effect of a Spin-Orbit Interaction on $M1$ Transitions

Sachs and Austern[18] have shown that the introduction of a velocity-dependent interaction will leave electric transition probabilities the same as calculated above, but that it will result in different magnetic transition probabilities. In particular, Jensen and Mayer[19] have shown that the introduction of a spin-orbit coupling for a single proton leads to finite $M1$ transition probabilities, both for $\Delta l=0$, and for $\Delta l=2$. In contrast, for a pure central potential, in the absence of spin-orbit coupling, no $M1$ transitions can occur, because either the $M1$ matrix element between initial and final state vanishes, or the two states are degenerate.

[18] R. G. Sachs and N. Austern, Phys. Rev. **81**, 705, 710 (1951).
[19] J. H. D. Jensen and M. G. Mayer, Phys. Rev. **85**, 1040 (1952).

With reasonable assumptions regarding the strength of spin-orbit coupling in nuclei ($\Delta E=2$ Mev for $A=100$, $l=4$),[20] Jensen and Mayer obtain $M^2=0.26$ for an $M1$ transition in an odd proton nucleus involving $\Delta l=2$.

e. Transitions between States of Several Particles

Strictly speaking, all nuclear states, except those of a single nucleon, are states of several particles. In most cases involving an odd number of particles, the spins of all but one particle couple to spin zero, and the state has the I of the odd particle.[8] These particular states will be referred to as "states of normal coupling." Such states behave in many ways like the corresponding states of a single particle, e.g., have the same spin and magnetic moment. It shall be assumed that the particles can be treated as independent; i.e., that the wave function of a state of several particles can be written as a sum of products of single particle wave functions.

States in which an odd number of identical particles couple to a spin different from the spin of the odd particle; e.g., $(g_{9/2})_{7/2+}{}^{3,5\text{ or }7}$ are also known to occur.[5,17] Such states are called here "states of abnormal coupling." Calculations of Kurath[21] and of Talmi[22] of energy levels for the different configurations of three identical particles in a $d_{5/2}$ or $f_{7/2}$ orbit indicate that the state of normal coupling is expected to be the ground state, provided the forces between particles are of short range, compared to the nuclear radius. However, for a range of forces comparable to the nuclear radius, Kurath and also Talmi find that the state $(j)_{j-1}{}^3$ can be the ground state. In view of these results, the existence of states of abnormal coupling should not be surprising.

The wave functions of individual nucleons in various quantum states shall be labeled $\psi_{j_k}{}^{m_k}$ and one writes symbolically for the properly antisymmetrized linear combination of single particle wave functions of n nucleons; i.e., for the Slater determinant,

$$\Psi=\psi_{j_1}{}^{m_1}\psi_{j_2}{}^{m_2}\cdots\psi_{j_n}{}^{m_n}. \quad (18)$$

The wave function for $2s$ identical particles, each of spin j, and differing only in m, which couple to a total spin 0, denoted by $\Psi(j^{2s})_0{}^0$ can be written as

$$\Psi(j^{2s})_0{}^0=\left(\sum_{m=\frac{1}{2}}^{j}(-1)^{j-m}\psi_j{}^m\psi_j{}^{-m}\right)^s\Bigg/\left[\frac{(j+\frac{1}{2})!s!}{(j+\frac{1}{2}-s)!}\right]^{\frac{1}{2}}. \quad (19)$$

It is easily seen that $\Psi(j^{2s})_0{}^0$ is an eigenfunction of I^2 and of I_z with the proper eigenvalues $I(I+1)=0$, $M=0$.

Any term containing the same value of m twice automatically vanishes. Since there are $(j+\frac{1}{2}!)/(s!j+\frac{1}{2}-s!)$ different nonvanishing terms, and each one occurs $s!$ times, the normalization of (19) is warranted.

[20] M. G. Mayer, Phys. Rev. **74**, 235 (1948).
[21] D. Kurath, Phys. Rev. **80**, 98 (1950).
[22] I. Talmi, Phys. Rev. **82**, 101 (1951).

A wave function of $2s+1$ particles, each of spin j, which couple to $I=j$, $M=j$ is given by

$$\Psi(j^{2s+1})_j{}^j = \psi_j{}^j\left(\sum_{m=\frac{1}{2}}^{j}(-1)^{j-m}\psi_j{}^m\psi_j{}^{-m}\right)^s \bigg/ \left[\frac{(j-\frac{1}{2})!s!}{(j-\frac{1}{2}-s)!}\right]^{\frac{1}{2}}. \quad (20)$$

For $j\leq 7/2$, these wave functions are the only ones which can be constructed.

The total wave function of a number of identical particles can thus be written, apart from normalization factors, as products of wave functions of pairs of identical particles coupling to a spin 0, and that of the wave function of the odd particle, if any. The total normalization factor is not equal to the product of the separate normalization factors, because many terms in the product can vanish.

The total wave function of n_a particles in an orbit a and n_b particles in a different orbit b is just the product of $\Psi(j_a{}^{n_a})$ and of $\Psi(j_b{}^{n_b})$, including the value of the normalization factor.

Using the wave functions (19) and (20), and assuming an interaction operator which can be written as a sum of terms, each of which acts on one particle at a time, and which is symmetric in all the particles (called a symmetric single-particle interaction operator),

$$O=\sum_{\text{particles } k} O_k, \quad (21)$$

one can evaluate the probability for transitions between states of normal coupling involving configurations of only protons (or neutrons).[23]

This transition probability is equal to the transition probability between the corresponding single particle states, multiplied by a factor denoted here by ρ, which depends on the number of particles in the two orbits involved in the transition and also on the spins of these orbits. We must consider two cases, according to whether the particle undergoing the transition is an odd one or an even one.

Let the initial state of spin j_a contain $2s_a+1$ particles in orbit a and $2s_b$ particles in orbit b, and the final state of spin j_b have $2s_a$ and $2s_b+1$ particles in orbits a and b, respectively.

[23] For many nuclear states, e.g. ground and low-lying isomeric states of heavy nuclei with $N\geq 50$ and also of some light nuclei, all orbits partially or completely filled by protons are also completely filled by neutrons. The isotopic spin of such a state is uniquely given as $(N-Z)/2$. A transition between two such states involving a particle jump between orbits a and b can be characterized as a proton (or neutron) transition if the orbits a and b are partially filled by protons (or neutrons).

According to the independent particle model, the transition probability between these two states is equal to the transition probability between two corresponding states of a hypothetical nucleus for which there are protons (or neutrons) only in orbits a and b. Thus, although in this section states consisting exclusively of protons (or neutrons) are considered, the results are applicable to heavy nuclei, in so far as the independent particle model is valid.

Then (odd particle jumping), ρ is given by

$$\rho=\frac{j_a+\frac{1}{2}-s_a}{j_a+\frac{1}{2}}\times\frac{j_b+\frac{1}{2}-s_b}{j_b+\frac{1}{2}}. \quad (22a)$$

Alternatively, let the initial state of spin j_a be composed of $2s_a-1$ particles and $2s_b$ particles in orbits a and b, and the final state of spin j_b contain $2s_a$ and $2s_b-1$ particles in these orbits.

Then (even particle jumping), ρ is given by

$$\rho=\frac{s_a}{j_a+\frac{1}{2}}\times\frac{s_b}{j_b+\frac{1}{2}}. \quad (22b)$$

According to Eqs. (22), the transition probability between states of partially filled orbits is less than that between the corresponding single particle states. Physically, this reduction comes about from the fact that a fraction of the substates, which would be available to the particle in the corresponding transition between single particle states, are not available in this case to the jumping particle, because they are already occupied.

It can also be shown that certain kinds of transitions cannot occur, according to an independent particle picture, assuming a symmetric single-particle interaction operator. These follow:

1. Transitions which require change of orbit for several particles.

This is an immediate consequence of the form of the interaction operator.

2. Transitions between a state of normal coupling and a state of abnormal coupling of only protons (or neutrons), involving a change of orbit for one particle.

Such a transition would require that several particles change their state, since not only is one particle changing its orbit, but at least two other particles must change from being lined up antiparallel to being lined up in a different way.

3. M1 transitions between two states of the same configuration, consisting exclusively of protons (or neutrons), and differing only in coupling and no change of orbit.

The transition matrix element is

$$\int\Psi_f(\textstyle\sum_k O_k)\Psi_i d\tau=c\int\Psi_f(\sum_k j_k)\Psi_i d\tau \quad (23)$$

which follows from the relation[24]

$$\int\psi_j{}^{m'}O\psi_j{}^m d\tau=c\int\psi_j{}^{m'}j\psi_j{}^m d\tau \quad (24)$$

[24] E. U. Condon and G. H. Shortley, *The Theory of Atomic Spectra* (University Press, Cambridge, England, 1951), first edition, p. 61.

where c is independent of the values of m and m'. Since the operator $I=\sum_k j_k$ does not mix wave functions of different states, the integral (23) must vanish.

f. Numerical Calculations of Radial Integrals

The radial integrals in Eqs. (10) depend on the form of the nuclear potential. It is questionable whether the single particle wave functions calculated for a central potential are good approximations to the real nuclear wave functions. However, the success of the independent particle model in explaining many observed phenomena[8] makes the attempt to use single particle wave functions at least plausible.

The radial integrals Eq. (12) have been evaluated for several cases:

1. *Wave Function Constant Inside Nucleus—Vanishing Outside*[13]

For EL radiation $\quad |(r/a)^L|_{fi}=3/(L+3)$. (25a)

For ML radiation $\quad |(r/a)^{L-1}|_{fi}=3/(L+2)$. (25b)

2. *Spherical Square Well Potential—No Spin-Orbit Coupling*

The wave functions are obtained by solving the eigenvalue problem for a square well (of radius a and depth V_0), i.e., by obtaining the solutions of the Schroedinger equation for $r\leq a$ and for $r\geq a$ and fitting the values of the wave functions and their derivatives at $r=a$.

Calculations of the radial integrals were made for several cases which represent typical isomeric transitions in nuclei. All states are assumed to be bound by 8 Mev. (V_0 is slightly different for initial and final state. The radius a has been taken as $1.5\times10^{-13}\,A^{\frac{1}{3}}$ cm.)

The results of these calculations are shown in Table II.

The radial integrals (in units of proper power of nuclear radius), are nearly independent of the size of the nucleus. This is a consequence of the fact that for the cases treated here, the binding energy is much smaller than the depth of the potential well.

The results of Table II suggest, in agreement with expectations, that values of radial integrals depend less on the particular nuclear model for large multipole orders of the radiation than for small multipole order, and that they do not depend strongly on the particular states nor on nuclear size for a given multipole order.

3. *Spherical Square Well Potential with Spin-Orbit Coupling*

The radial integral was evaluated for a square well potential with an additional spin-orbit interaction given by

$$-K\frac{1}{r}\frac{dV}{dr}(\mathbf{l}\cdot\boldsymbol{\sigma}) \qquad (26)$$

TABLE II. Calculated values of radial integral.

Transition[a]	Cases of interest	Radial integral $\|(r/a)^{L \text{ or } L-1}\|_{fi}$ for EL or ML: Square well $A=100$	Square well $A=140$	Constant wave function
$1h-3s$	$E5\ h_{11/2}\rightarrow s_{1/2}$	0.43	0.40	0.38
$1g-2p$	$M4\ g_{9/2}\rightarrow p_{1/2}$	0.45	0.43	0.50
$1h-2d$	$M4\ h_{11/2}\rightarrow d_{3/2}$ $E3\ h_{11/2}\rightarrow d_{5/2}$	0.43	0.42	0.50
$2d-3s$	$E2\ d_{5/2}\rightarrow s_{1/2}$	0.52	0.50	0.60

[a] For each state, the number denotes radial quantum number n (number of radial nodes +1), letter corresponds to orbital quantum number l.

(which is infinite at $r=a$ and vanishes everywhere else), for the special case of an $M4$ transition between a $1g_{9/2}$ and $2p_{1/2}$ state in In^{113}.

The value of K and of well depth V_0 were so chosen that both initial and final state are bound by 8 Mev. The radial wave functions are assumed to be of the same kind as those of a square well, but the boundary conditions require a finite discontinuity in the derivative at $r=a$. The value of the radial integral for this case is 0.53 compared to 0.44 for the same transition in a square well potential without a spin-orbit interaction.

g. Summary of Theoretical Results

The lifetime for a nuclear isomeric state for gamma-decay can be written as follows (Sec. IIa):

$$\tau_{\gamma EL \atop {\Delta I=L \atop \Delta l=L}}=\left[\frac{2(L+1)}{L}\frac{1}{(2L+1!!)^2}\omega\frac{e^2}{\hbar c}\left(\frac{\omega a}{c}\right)^{2L}SM_{EL}^2\right]^{-1}; \qquad (9a)$$

$$\tau_{\gamma ML \atop {\Delta I=L \atop \Delta l=L-1}}=\left[\frac{2(L+1)}{L}\frac{1}{(2L+1!!)^2}\omega \times\frac{e^2}{\hbar c}\left(\frac{\omega a}{c}\right)^{2L}\left(\frac{h}{Mca}\right)^2 SM_{ML}^2\right]^{-1}. \qquad (9b)$$

For transitions for which $I_i=I_f+L$,

$$S(I_i,L,I_f)=\frac{(I_i-\frac{1}{2})!}{(2I_i)!!}\frac{2L+1!!}{L!}\frac{(2I_f)!!}{(I_f-\frac{1}{2})!}, \qquad (14a)$$

and a simple statistical weight argument gives S for transitions for which $I_i=I_f-L$.

In the later discussion of $M4$ transitions (Sec. IIIa), the following values of S will be of particular interest:

$$S(9/2,4,\tfrac{1}{2})=1; \qquad (27a)$$

$$S(\tfrac{1}{2},4,9/2)=5; \qquad (27b)$$

$$S(11/2,4,\tfrac{3}{2})=15/11; \qquad (27c)$$

$$S(13/2,4,5/2)=225/143. \qquad (27d)$$

The value of M^2 depends on the particular model of the nucleus. For a single proton (Sec. IIa),

$$M_{EL}^2 = |(r/a)^L|_{fi}^2; \tag{10a}$$

$$M_{ML}^2 = \left(\mu_P L - \frac{L}{L+1}\right)^2 \left|\left(\frac{r}{a}\right)^{L-1}\right|_{fi}^2. \tag{10b}$$

For a single neutron (Sec. IIc),

$$M_{EL}^2 = \left(\frac{Z}{A^L}\right)^2 \left|\left(\frac{r}{a}\right)^{L}\right|_{fi}^2, \tag{28a}$$

or of order $(\hbar\omega/Mc^2)^2$, whichever is larger.

For $E3$ transitions of energy 100 kev, $M^2 \sim 10^{-8}$.

$$M_{ML}^2 = (\mu_N L)^2 |(r/a)^{L-1}|_{fi}^2. \tag{28b}$$

Theoretical values of M^2 for $M4$ transitions square-well potential, $A=100$ (Secs. IIf, IIIa), follow:

	Transition	Group	M_{M4}^2	
proton	$1g_{9/2} \leftrightarrow 2p_{1/2}$	$(4P)$	22	(29a)
neutron	$1g_{9/2} \leftrightarrow 2p_{1/2}$	$(4N)$	11.8	(29b)
neutron	$h_{11/2} \leftrightarrow 2d_{3/2}$	$(5N)$	10.8	(29c)

For transitions between states of normal coupling, involving configurations of only protons (or neutrons), calculations based on an independent particle model (Sec. IIe), give as result that M^2 is equal to M^2 for the corresponding transition between pure single particle states, times a factor ρ.

For $[(j_a)^{2s_a+1}(j_b)^{2s_b}]_{j_a} \rightarrow [(j_a)^{2s_a}(j_b)^{2s_b+1}]_{j_b}$,

$$\rho = \frac{j_a+\frac{1}{2}-s_a}{j_a+\frac{1}{2}} \times \frac{j_b+\frac{1}{2}-s_b}{j_b+\frac{1}{2}}. \tag{22a}$$

For $[(j_a)^{2s_a-1}(j_b)^{2s_b}]_{J_a} \rightarrow [(j_a)^{2s_a}(j_b)^{2s_b-1}]_{J_b}$,

$$\rho = \frac{s_a}{j_a+\frac{1}{2}} \times \frac{s_b}{j_b+1}. \tag{22b}$$

TABLE III. Distribution of $\log M^2$ values for $M4$ transitions in odd-A nuclei; subscript denotes group of transition.

Pt_{6N}^{195} Ba_{5N}^{133} Nb_{4P}^{97}	Hg_{6N}^{199} Hg_{6N}^{197} Pt_{6N}^{197} Xe_{5N}^{129} Zn_{4N}^{69}	Ba_{5N}^{137} Ba_{5N}^{135} Xe_{5N}^{131} Te_{5N}^{129} In_{4P}^{115} In_{4P}^{113} Y_{4P}^{91}	Pb_{6N}^{207} Xe_{5N}^{133} Te_{5N}^{127} Y_{4P}^{87} Sr_{4N}^{87} Sr_{4N}^{85}	Xe_{5N}^{135} Te_{5N}^{125} Te_{5N}^{123} Te_{5N}^{121} Sn_{5N}^{117} Tc_{4P}^{99} Tc_{4P}^{97} Tc_{4P}^{95} Nb_{4P}^{95} Zr_{4N}^{89} Kr_{4N}^{85}
$\bar{9}.7$[a]	$\bar{9}.8$	$\bar{9}.9$	0.0	0.1
	Sn_{5N}^{119}	Y_{4P}^{89}	Te_{5N}^{131}	Nb_{4P}^{91}
	0.2	0.3	0.4	0.9
		$\log_{10}M^2$		

[a] $\bar{9}.7$ denotes $9.7-10$.

Example: The transition $[p_{\frac{1}{2}}(g_{9/2})^2]_{\frac{1}{2}-} \rightarrow [(g_{9/2})^3]_{9/2+}$ has a transition probability 4/5 as large as $p_{\frac{1}{2}} \rightarrow g_{9/2}$ transition. A $[p_{\frac{1}{2}}(g_{9/2})^2]_{\frac{1}{2}-} \rightarrow [(p_{\frac{1}{2}})^2(g_{9/2})]_{9/2+}$ transition has a transition probability only 1/5 as large as the transition between single particle states. Thus the three transitions are all between $\frac{1}{2}-$ and $9/2+$ states yet the transition probabilities are different.

Transitions requiring change of orbit for several particles cannot occur according to an independent particle model (Sec. IIe); neither can transitions between a state of normal coupling and one of abnormal coupling of only protons (or neutrons), involving a change of orbit for one particle, e.g., $[p_{\frac{1}{2}}(g_{9/2})^2]_{\frac{1}{2}-} \rightarrow [(g_{9/2})^3]_{7/2+}$.

$M1$ transitions between two states of the same configuration of only protons (or neutrons), and differing only in coupling, and involving no change of orbit, e.g., $[(g_{9/2})^3]_{9/2+} \rightarrow [(g_{9/2})^3]_{7/2+}$, cannot occur either.

III. SOME INTERPRETATIONS—VALUES OF M^2

Values of empirical squares of matrix elements M^2, obtained by comparing the experimental lifetime, corrected for internal conversion, with the theoretical lifetime of Eqs. (9) or (15), show some interesting regularities, which may have significance in connection with nuclear shell structure.[8]

A comprehensive summary of the classification of isomeric transitions as well as a discussion of values of empirical matrix elements has been given by Goldhaber and Sunyar.[5] Some of their conclusions were derived independently by the author.[17] It is found that most magnetic transitions have empirical M^2 values of order 1, while most electric transitions (except some $E2$), have smaller M^2 values, generally of order 10^{-3}, resulting in comparable lifetimes for EL and ML transitions of the same energy.

However, some additional results of an analysis of M^2 values, which were not discussed in the above papers, can be mentioned at this point; *viz.*, $M4$ transitions in odd-A nuclei, $E3$ transitions in odd-A nuclei, and the $M1$ transition in Li^7.

a. $M4$ Transitions in Odd-A Nuclei

$M4$ transitions are found precisely where they are expected according to the j-j coupling shell model.[8] These transitions may be divided into four groups, according to the number of the shell in which they occur (near the end), and according to whether N or Z is odd. The shells containing up to 50, 82, and 126 particles are denoted by 4, 5, and 6 (corresponding transitions: $g_{9/2} \leftrightarrow p_{1/2}$, $h_{11/2} \rightarrow d_{3/2}$, and $i_{13/2} \rightarrow f_{5/2}$), respectively, in accordance with the order of filling of shells in the j-j coupling model.

Empirical values of M^2 for $M4$ transitions (calculated with the aid of Eqs. (10b) and (27), are in good agreement with the estimates of Weisskopf,[13] $M^2=1$, and show surprising lack of scattering, as can be seen from Table III. The average and scatter of $\log M^2$ values for

TABLE IV. Average and scatter of $\log M^2$ values for various groups of $M4$ transitions in odd-A nuclei.

Group	Number of cases	Average	Scatter[a]
$4P\ g_{9/2}-p_{1/2}$	11	0.08	0.32
$4N\ g_{9/2}-p_{1/2}$	5	$\bar{9}.99$	0.09
$5N\ h_{11/2}-d_{3/2}$	15	0.03	0.16
$6N\ i_{13/2}-f_{5/2}$	5	$\bar{9}.83$	0.10
Total odd$-Z$	11	0.08	0.32
Total odd$-N$	25	$\bar{9}.98$	0.16
Total	36	0.01	0.23

[a] As measured by root mean square deviation.

$M4$ transitions of the various groups ($4P$, $4N$, $5N$, and $6N$) are shown in Table IV. It is seen that the average value of $\log M^2$ is nearly identical for the $4P$, $4N$, and $5N$ groups, and slightly smaller for the small $6N$ group. Goldhaber and Sunyar[5,25] have calculated M^2 by putting $S=1/(2I_i+1)$, rather than values given by Eq. (27), their idea probably being that lifetimes should be proportional to the number of initial m-substates. Owing to this difference, they find a considerably larger average value for M^2 in the $5N$ group than in the $4N$ group. As the statistical factors used here, Eqs. (14, 27) are uniquely given by integration of the angular parts of the nuclear wave functions, their adoption seems more realistic than the use of the factors proposed by Goldhaber and Sunyar.[5]

Theoretical values of M^2 for $M4$ transitions (assuming a single proton or neutron in a spherical square well, nucleus of $A=100$, $a=1.5\times10^{-13}\ A^{\frac{1}{3}}$ cm), are given by Eqs. (29) for $4P$, $4N$, and $5N$ transitions.

On the basis of a single particle model, one would thus expect M^2 values for odd proton nuclei as well as odd neutron nuclei to show very little scattering, and also very little difference between the $4N$ and $5N$ groups.

One would, however, expect a considerable difference in M^2 values between odd proton and odd neutron nuclei, with the former expected to be about twice as large as the latter. No such difference between M^2 values is found experimentally, as far as can be ascertained from the data available to date. According to Eqs. (29), M^2 values predicted by the single particle model are of order 10, while empirical M^2 values are of order 1. One might attempt to reduce this apparent discrepancy by modifying the theoretical predictions; e.g., by substituting for μ_P and μ_N not the magnetic moment of a free particle, but the effective intrinsic magnetic moment for an odd particle, which can be inferred from a magnetic moment measurement of the ground state, on the assumption that the spin and magnetic moment of the nucleus is determined by that of the odd particle.[26] This would give values of theoretical M^2 lower than the values of Eqs. (29). However, a large amount of scattering of theoretical M^2 values would now result, due to scattering of values of μ_P and μ_N for ground states.

Furthermore, the difference between theoretical M^2 values for odd-proton and odd-neutron transitions would still exist. In any case, it is questionable whether any relation exists between matrix elements for $M4$ transitions and static $M1$ momenta of ground states, since the "intrinsic magnetic moments," which are assumed to influence the magnitude of these quantities, may be quite different from each other.

Empirical values of $\log M^2$ as function of neutron number (Fig. 1) show some interesting regularities. For $4P$ isomers, $\log M^2$ as function of N has a maximum at $N=50$, while for $5N$ isomers, it tends to increase as function of $N(N<82)$ and decreases as function of $Z(Z>50)$. In general, M^2 values appear to be larger, the more closely the nucleus can be represented by closed shells. The apparent tendency of empirical M^2 to be larger for nearly closed shell nuclei than for others receives some support from theoretical values of M^2 calculated for transitions between states containing several particles on the basis of an independent particle model [Sec. IIe, Eqs. (22)].

The M^2 values for transitions between states of normal coupling are, in general, found to be smaller than those for the corresponding true single particle states.

b. $E3$ Transitions in Odd-A Nuclei

Most known $E3$ transitions in odd-A nuclei are either in odd neutron nuclei or between $p_{1/2}$ and $7/2+$ states (the latter for odd particle number 43, 45, or 47). For a single neutron transition, the value of M^2, according to an independent particle model, should be of order $(Z/A^L)^2$ because of the recoil of the core, or of order $(\hbar\omega/Mc^2)^2$ because of the intrinsic magnetic moment of the $(\hbar\omega/Mc^2)^2$ neutron, whichever is the larger (Sec. IIc); i.e., be of order 10^{-8} for $L=3$, $A=100$, $E=100$ kev. The empirical value of M^2 for the $E3$ transition in $Cd^{111}(h_{11/2}\rightarrow d_{5/2}$, energy 149 kev) is 4.5×10^{-6}.

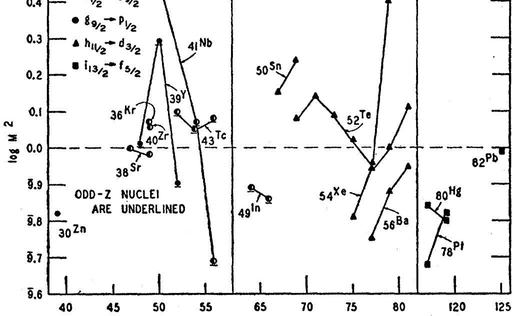

FIG. 1. Values of $\log M^2$ for $M4$ transitions *versus* neutron number (odd-A nuclei).

[25] The term "$|M|^2$" used by Goldhaber and Sunyar is the same as SM^2 used in this paper, while their "$|M'|^2$" is equivalent to M^2 used here, the latter normalized to an average value of 1.
[26] F. Bloch, Phys. Rev. **83**, 839 (1951).

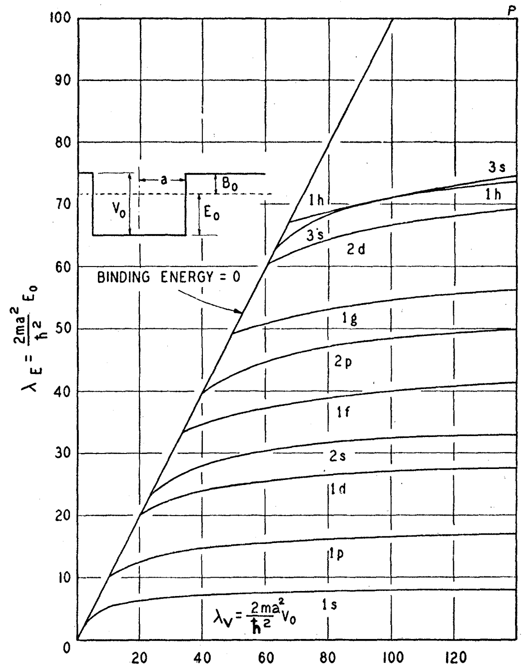

FIG. 2. Energy levels for a spherical square well.

For a transition between $p_{1/2}$ and $7/2+$ states, M^2 vanishes according to the independent particle model (Sec. IIe). It is thus not surprising that the empirical values of M^2 for $E3$ transitions are small and show considerable scattering, as shown in Table V.

The size of the M^2 can be taken as a measure of the deviations from a pure independent particle model. As it is likely that the size of these deviations will probably be different nuclei, the variation of $\log M^2$ values shown in Table V can be understood in this picture. In fact, it can be seen that $\log M^2$ decreases as function of N for N between 43 and 47; i.e., as the neutron number approaches 50. This is not surprising, since one would expect greater validity of the independent particle model for a given nucleus, the more nearly it is a closed shell nucleus.

c. $M1$ Transition in Li^7

The $\frac{1}{2}$-$\rightarrow\frac{3}{2}$-transition in Li^7 (energy 479 kev), is the only one of the $M1$ group known for which the independent particle model (including the effect of spin-orbit coupling) predicts matrix elements of order 1.

The theoretical M^2 value according to this model is $(\mu_P-\frac{1}{2})^2=5.25$ for a free proton undergoing a $p_{\frac{1}{2}}\rightarrow p_{\frac{3}{2}}$ transition. According to calculations similar to the ones described in section IIe, M^2 is 2.7 for a $[(p_{\frac{3}{2}})^3]_{\frac{1}{2}-}\rightarrow[(p_{\frac{3}{2}})^3]_{\frac{3}{2}-}$ transition, and 1.0 for a $[(p_{\frac{3}{2}})^2(p_{\frac{1}{2}})]_{\frac{1}{2}-}\rightarrow[(p_{\frac{3}{2}})^3]_{\frac{3}{2}-}$ transition, for a system of two neutrons and one proton with isotopic spin $\frac{1}{2}$ in both initial and final state.

The empirical value of M^2 is 7.6. The fact that the empirical value of M^2 is larger than the theoretical M^2 for any of the configurations mentioned, is somewhat surprising, but is probably not significant, due to the questionable validity of the assumptions made in the calculations.

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TABLE V. Empirical $\log M^2$ values for some $E3$ transitions between $p_{1/2}$ and $7/2+$ states in odd-N nuclei.

	Z	34	36
N 43		$\bar{7}.4^{a}$	$\bar{6}.6$
45		$\bar{6}.6$	$\bar{6}.5$
47		$\bar{5}.8$	$\bar{6}.2$

[a] $\bar{7}.4$ denotes $7.4-10$.

APPENDIX. ENERGY LEVELS FOR A SPHERICAL SQUARE-WELL POTENTIAL

To evaluate the radial wave functions (Sec. IIf), it was first necessary to obtain energy levels. A graph of energy levels is given here in the hope that it may be useful for related problems of energy levels in a spherical square-well potential. Assume,

$$\begin{matrix}\lambda_E\\ \lambda_B\\ \lambda_V\end{matrix} = \frac{2Ma^2}{\hbar^2}\begin{Bmatrix}E_0\\ B_0\\ V_0\end{Bmatrix}, \qquad \begin{matrix}(30a)\\(30b)\\(30c)\end{matrix}$$

where a = radius of well, V_0 = depth of potential, E_0 = kinetic energy of particle in nucleus, and B_0 = binding energy $(E_0+B_0=V_0)$. Specifically, assuming $a=1.5\times10^{-13}\times A^{\frac{1}{3}}$ cm and taking energies in Mev, we obtain

$$2Ma^2/\hbar^2=0.1084A^{\frac{2}{3}}\text{ Mev}^{-1}. \qquad (31)$$

The energy value solutions of the Schroedinger equation for states with radial quantum numbers n (number of radial nodes $+1$), and orbital angular momentum numbers l are plotted as function of λ_V in Fig. 2. Using Fig. 2 it is possible to obtain the position of energy levels for square wells for a wide range of radii and well depths.