

THE
LONDON, EDINBURGH, AND DUBLIN
PHILOSOPHICAL MAGAZINE
AND
JOURNAL OF SCIENCE.

[SEVENTH SERIES.]

JANUARY 1927.

I. *The Kinematics of an Electron with an Axis.*
By L. H. THOMAS, B.A., Trinity College, Cambridge*.

Introduction.

IT seems that Abraham † was the first to consider in any detail an electron with an axis. Many have since considered spinning electrons, ring electrons, and the like. Compton ‡ in particular suggested a quantized spin for the electron. It remained for Uhlenbeck and Goudsmit § to show how this idea can be used to explain the anomalous Zeeman effect. The assumptions they had to make seemed to lead to optical and relativity doublet separations twice as large as those observed ||.

The purpose of the following paper, which contains the results mentioned in my recent letter to 'Nature' ¶, is to investigate the kinematics of an electron with an axis on the basis of the restricted theory of relativity. The main fact used is that the combination of two "Lorentz transformations without rotation" in general is not of the same form but is equivalent to a Lorentz transformation with a rotation.

The physical interest of the result obtained is that it shows that Uhlenbeck and Goudsmit's assumptions really lead to

* Communicated by Prof. N. Bohr, Ph.D., LL.D.

† M. Abraham, *Annalen der Physik*, x. p. 105 (1903).

‡ A. K. Compton, *Journ. Frankl. Inst.* Aug. 1921, p. 145.

§ Uhlenbeck and Goudsmit, *Naturwissenschaften*, Nov. 20, 1925, p. 953; 'Nature,' Feb. 20, 1926, p. 264.

|| See also: Urly and Bichowski, *Proc. Nat. Acad. of Sciences*, xii. p. 80 (1926).

¶ 'Nature,' April 10, 1926, p. 514.

the correct doublet separation at the same time as the anomalous Zeeman effect when the problem is treated by the new quantum mechanics*. These explanations do not seem to require anything more of the extra terms in the equations of motion of the electron than that its axis should precess about a magnetic field \mathbf{H} with angular velocity $(e/mc)\mathbf{H}$, that in revolution in an orbit there be some secularly conserved angular momentum, and that the contribution of the electron to this angular momentum be $h/4\pi$.

The above assumptions are not, however, sufficient for more than the first order doublet separation. The complete Sommerfeld formula would seem to require a more complete specification of the extra terms in the equations of motion.

Summary.

In the first four sections of this paper the notation used is explained, the relativity kinematics involved is discussed, and a first approximation to the equations of motion of the electron is considered. The change in the direction of its axis is given by

$$\frac{d\mathbf{w}}{ds} = \left[\left\{ \frac{e}{mc} \mathbf{H} - \frac{e}{mc^2} \frac{\beta}{1+\beta} [\mathbf{v} \times \mathbf{E}] \right\} \times \mathbf{w} \right]. \quad (4.122)$$

The Abraham spinning electron is discussed briefly. In spite of its inadequacy it is interesting as showing that the assumptions made are not unreasonable.

In the sixth section the secular changes in an electronic orbit are discussed. The equations obtained are

$$\frac{d\Omega}{dt} = \left[\left(\frac{e}{mc} \mathbf{H} + \frac{\sigma}{K} \mathbf{K} \right) \times \Omega \right], \quad \left. \vphantom{\frac{d\Omega}{dt}} \right\} \quad . \quad . \quad (6.71)$$

$$\frac{d\mathbf{K}}{dt} = \left[\left(\frac{e}{2mc} \mathbf{H} + \frac{\sigma}{K} \Omega \right) \times \mathbf{K} \right]. \quad \left. \vphantom{\frac{d\mathbf{K}}{dt}} \right\} \quad . \quad . \quad (6.72)$$

In the seventh section a correspondence principle argument is developed giving approximate Zeeman effect and doublet term values and the Heisenberg theory modification of these values is stated †.

Finally, a summary of the reasoning by which the Uhlenbeck-Goudsmit theory connects the Zeeman effect and doublet separation is given ‡.

* W. Heisenberg and P. Jordan, *Zeit. für Phys.*, xxxvii. p. 266 (1926).

† The Heisenberg theory term values, which were very kindly supplied to me by Dr. Heisenberg, are taken from the paper by him and Professor Jordan (*loc. cit.*).

‡ Uhlenbeck and Goudsmit, *loc. cit.*

1. Notation.

In the sequel it is convenient to use in some places ordinary vector notation, letters in heavy type denoting vectors, the same letters with suffixes 1, 2, 3 their components, and in other places tensor notation, with the usual summation convention, e. g. $g_{\mu\nu} dx^\nu$ short for $\sum_\nu g_{\mu\nu} dx^\nu$. It will be well, therefore, to connect the two notations here, as well as to state the units that are adopted, especially as these differ from those used by Eddington*.

The position-vector \mathbf{r} of a particle and the time t at which it is at that position are connected with its four coordinates x^1, x^2, x^3, x^4 by

$$\left. \begin{aligned} \mathbf{r} &= (x^1, x^2, x^3), \\ t &= x^4, \end{aligned} \right\} \dots \dots \dots (1.01)$$

\mathbf{r} is in centimetres, t in seconds. s , the "proper time" for the particle, is given by the relation

$$c^2 ds^2 = c^2 dt^2 - d\mathbf{r}^2 = g_{\mu\nu} dx^\mu dx^\nu, \dots \dots (1.02)$$

where the components of $g_{\mu\nu}$, the "fundamental tensor" of relativity theory, have here, as throughout this paper, their values for "Galilean coordinates," $g_{11} = g_{22} = g_{33} = -1$, $g_{44} = c^2$, the remaining components vanishing, and where c is the velocity of light.

Let

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}, \dots \dots \dots (1.11)$$

$$\beta = \left(1 - \frac{\mathbf{v}^2}{c^2}\right)^{-\frac{1}{2}} = \left(1 + \frac{1}{c^2} \left(\frac{d\mathbf{r}}{ds}\right)^2\right)^{\frac{1}{2}} = \frac{dt}{ds}, \dots (1.12)$$

so that

$$\left. \begin{aligned} \beta \mathbf{v} &= \left(\frac{dx^1}{ds}, \frac{dx^2}{ds}, \frac{dx^3}{ds}\right), \\ \beta &= \frac{dx^4}{ds}. \end{aligned} \right\} \dots \dots \dots (1.13)$$

The electric intensity \mathbf{E} and scalar potential ϕ are in electrostatic units, the magnetic intensity \mathbf{H} and vector

* A. S. Eddington, 'The Mathematical Theory of Relativity,' p. 170.

potential \mathbf{A} in electromagnetic units, so that

$$(\nabla \cdot \mathbf{A}) + \frac{1}{c} \dot{\phi} = 0, \quad (1.21)$$

$$\left. \begin{aligned} \mathbf{E} &= -\nabla \phi - \frac{1}{c} \dot{\mathbf{A}}, \\ \mathbf{H} &= [\nabla \times \mathbf{A}], \end{aligned} \right\} (1.22)$$

and in empty space the field equations are

$$\left. \begin{aligned} (\nabla \cdot \mathbf{H}) &= 0, \\ [\nabla \times \mathbf{E}] + \frac{1}{c} \dot{\mathbf{H}} &= 0, \end{aligned} \right\} (1.23)$$

$$\left. \begin{aligned} (\nabla \cdot \mathbf{E}) &= 0, \\ [\nabla \times \mathbf{H}] - \frac{1}{c} \dot{\mathbf{E}} &= 0. \end{aligned} \right\} (1.24)$$

The four-vector k_μ and anti-symmetrical tensor $F_{\mu\nu}$ of the electromagnetic field are given by

$$\left. \begin{aligned} -\mathbf{A} &= (k_1, k_2, k_3), \\ c\phi &= k_4, \end{aligned} \right\} (1.31)$$

$$\left. \begin{aligned} c\mathbf{E} &= (F_{14}, F_{24}, F_{34}), \\ \mathbf{H} &= (F_{23}, F_{31}, F_{12}), \end{aligned} \right\} (1.32)$$

$F_{\mu\nu} = -F_{\nu\mu}$ and $F^{\mu\nu} = g^{\mu\sigma} F_{\sigma\nu}$, etc. as usual.

Equations (1.21), (1.22), (1.23), (1.24) become (for Galilean coordinates), as usual,

$$\frac{\partial k_\mu}{\partial x_\mu} = 0, \quad (1.41)$$

$$F_{\mu\nu} = \frac{\partial k_\mu}{\partial x_\nu} - \frac{\partial k_\nu}{\partial x_\mu}, \quad (1.42)$$

$$\frac{\partial F_{\mu\nu}}{\partial x_\sigma} + \frac{\partial F_{\nu\sigma}}{\partial x_\mu} + \frac{\partial F_{\sigma\mu}}{\partial x_\nu} = 0, \quad (1.43)$$

$$\frac{\partial F^{\mu\nu}}{\partial x_\nu} = 0. \quad (1.44)$$

If

$$\left. \begin{aligned} -c\mathbf{H} &= (\Phi_{14}, \Phi_{24}, \Phi_{34}), \\ \mathbf{E} &= (\Phi_{23}, \Phi_{31}, \Phi_{12}), \\ \Phi_{\mu\nu} &= -\Phi_{\nu\mu}. \end{aligned} \right\} (1.5)$$

$\Phi_{\mu\nu}$ is also a tensor.

The usual equations of motion of a particle of charge ϵ , rest-mass m take the form

$$\frac{d}{dt}(m\beta\mathbf{v}) = \epsilon \left\{ \mathbf{E} + \frac{[\mathbf{v} \times \mathbf{H}]}{c} \right\}, \quad \dots \quad (1.71)$$

equivalent to

$$\left. \begin{aligned} \frac{d}{ds} \left(m \frac{d\mathbf{r}}{ds} \right) &= \epsilon \left\{ \frac{dt}{ds} \mathbf{E} + \frac{1}{c} \left[\frac{d\mathbf{r}}{ds} \times \mathbf{H} \right] \right\}, \\ \frac{d}{ds} \left(m c^2 \frac{dt}{ds} \right) &= \epsilon \left(\frac{d\mathbf{r}}{ds} \cdot \mathbf{E} \right), \end{aligned} \right\} \quad (1.72)$$

or

$$\frac{d}{ds} \left(m \frac{dx^\mu}{ds} \right) = - \frac{\epsilon}{c} F^\mu{}_\nu \frac{dx^\nu}{ds} \dots \dots \dots (1.73)$$

2. Lorentz Transformations.

The idea of a Lorentz transformation with velocity \mathbf{v} unrestricted in direction is made precise by the following definition. The coordinate system (\mathbf{r}', t') is said to be obtained from the system (\mathbf{r}, t) by a Lorentz transformation with velocity \mathbf{v} without rotation when the equations of transformation take the form

$$\left. \begin{aligned} \mathbf{r}' &= \mathbf{r} + (\beta - 1) \frac{(\mathbf{r} \cdot \mathbf{v})}{v^2} \mathbf{v} - \beta \mathbf{v} t, \\ t' &= \beta \left\{ t - \frac{(\mathbf{r} \cdot \mathbf{v})}{c^2} \right\}, \end{aligned} \right\} \dots \dots (2.1)$$

where $\beta = (1 - v^2/c^2)^{-\frac{1}{2}}$. Here, of course, \mathbf{v} has nothing to do with the velocity of a particle as defined in (1.11), but is a constant of the transformation.

It is well known that all transformations leaving the form of (1.02) unaltered are combinations of a translation, including change of zero-point of the time, a space rotation, and a transformation of the form (2.1), while conversely any such combination leaves the form of (1.02) unaltered. If the order in which these are taken is fixed, the resolution is unique.

The combination of two successive transformations of the form (2.1) is not in general of the form (2.1) but can be resolved into a transformation of that form together with a rotation. In particular it follows from the theorem of Roderigues and Hamilton, applied to (\mathbf{r}, ict) space or can easily be shown directly, (*cf.* (3.6) below), that the resultant

of such a Lorentz transformation with velocity $-\mathbf{v}$ followed by one with velocity $\mathbf{v} + \delta\mathbf{v}$, where $\delta\mathbf{v}$ is infinitesimal, resolves into a Lorentz transformation with infinitesimal velocity

$$\left. \begin{aligned} & \beta \left\{ \delta\mathbf{v} + (\beta - 1) \frac{(\mathbf{v} \cdot \delta\mathbf{v})}{v^2} \mathbf{v} \right\} \\ & \text{together with an infinitesimal rotation} \\ & (\beta - 1) \frac{[\mathbf{v} \times \delta\mathbf{v}]}{v^2} \end{aligned} \right\} \quad \dots \quad (2.2)$$

The transformation of the electromagnetic field corresponding to (2.1) is

$$\left. \begin{aligned} \mathbf{E}' &= \beta \left\{ \mathbf{E} + \frac{[\mathbf{v} \times \mathbf{H}]}{c} \right\} + (1 - \beta) \frac{(\mathbf{E} \cdot \mathbf{v})}{v^2} \mathbf{v}, \\ \mathbf{H}' &= \beta \left\{ \mathbf{H} - \frac{[\mathbf{v} \times \mathbf{E}]}{c} \right\} + (1 - \beta) \frac{(\mathbf{H} \cdot \mathbf{v})}{v^2} \mathbf{v}. \end{aligned} \right\} \quad (2.3)$$

3. The kinematical description of the motion of an electron.

Suppose first that the motion of the electron can be described, to a sufficient approximation, by giving the position at any time in (\mathbf{r}, t) of a system of coordinates moving with the electron; that is, by giving coordinate systems (ρ, τ) in which, at different times in (\mathbf{r}, t) , the electron is instantaneously at rest at $\rho=0, \tau=0$, and which differ successively from one another by the rotation, supposed definite, which the electron undergoes, as well as by that change of origin and time (τ) direction necessary for the electron still to be instantaneously at rest at $\rho=0, \tau=0$ in (ρ, τ) . This will fix the system (ρ_1, τ_1) for $t=t_1$ in (\mathbf{r}, t) in terms of (ρ_0, τ_0) for $t=t_0$.

At $t=t_0$ let the electron have position \mathbf{r}_0 and velocity \mathbf{v}_0 , with $\beta_0 = (1 - v_0^2/c^2)^{-\frac{1}{2}}$, in (\mathbf{r}, t) . Then, by (2.1), that definite system of coordinates (\mathbf{R}_0, T_0) in which the electron is instantaneously at rest at the origin and which is obtained from (\mathbf{r}, t) by a translation and a Lorentz transformation without rotation is given by

$$\left. \begin{aligned} \mathbf{R}_0 &= \mathbf{r} - \mathbf{r}_0 + (\beta_0 - 1) \frac{(\mathbf{r} - \mathbf{r}_0 \cdot \mathbf{v}_0)}{v_0^2} \mathbf{v}_0 - \beta_0 \mathbf{v}_0 (t - t_0), \\ T_0 &= \beta_0 \left\{ t - t_0 - \frac{(\mathbf{r} - \mathbf{r}_0 \cdot \mathbf{v}_0)}{c^2} \right\} \end{aligned} \right\} \quad (3.1)$$

The system (ρ_0, τ_0) is specified by Eulerian angles $(\phi_0, \theta_0, \psi_0)$ giving its angular position in (\mathbf{R}_0, T_0) , so that

$$\left. \begin{aligned} \rho_{01} &= R_{01} (\cos \phi_0 \cos \theta_0 \cos \psi_0 - \sin \phi_0 \sin \psi_0) \\ &\quad + R_{02} (\sin \phi_0 \cos \theta_0 \cos \psi_0 + \cos \phi_0 \sin \psi_0) \\ &\quad \quad \quad - R_{03} \sin \theta_0 \cos \psi_0, \\ \rho_{02} &= R_{01} (-\cos \phi_0 \cos \theta_0 \sin \psi_0 - \sin \phi_0 \cos \psi_0) \\ &\quad + R_{02} (-\sin \phi_0 \cos \theta_0 \sin \psi_0 + \cos \phi_0 \cos \psi_0) \\ &\quad \quad \quad + R_{03} \sin \theta_0 \sin \psi_0, \\ \rho_{03} &= R_{01} \cos \phi_0 \sin \theta_0 \\ &\quad + R_{02} \sin \phi_0 \sin \theta_0 \quad \quad \quad + R_{03} \cos \theta_0, \end{aligned} \right\}$$

which will be abbreviated as

$$\rho_{0\alpha} = \zeta_{0\alpha b} R_{0b}, \quad . \quad . \quad . \quad (3.21)$$

while $\tau_0 = T_0$.

The system (ρ_1, τ_1) for $t = t_1 = t_0 + dt_0$ in (\mathbf{r}, t) is specified in the same way in terms of $\mathbf{r}_1 = \mathbf{r}_0 + d\mathbf{r}_0$, $\mathbf{v}_1 = \mathbf{v}_0 + d\mathbf{v}_0$, etc. Let

$$\left. \begin{aligned} d\zeta_0 &= (d\psi_0 \sin \theta_0 \cos \phi_0 - d\theta_0 \sin \phi_0, \\ &\quad d\psi_0 \sin \theta_0 \sin \phi_0 + d\theta_0 \cos \phi_0, \quad d\phi_0 + d\psi_0 \cos \theta_0). \end{aligned} \right\} (3.22)$$

Since the electron is instantaneously at rest at the origin in both systems, the infinitesimal transformation from (ρ_0, τ_0) to (ρ_1, τ_1) must be of the form

$$\left. \begin{aligned} \rho_1 - \rho_0 &= \left\{ [\rho_0 \times \mathbf{l}_0] - \tau_0 \mathbf{n}_0 \right\} ds_0, \\ \tau_1 - \tau_0 &= - \left\{ 1 + \frac{(\rho_0 \cdot \mathbf{n}_0)}{c^2} \right\} ds_0, \end{aligned} \right\} . \quad . \quad (3.23)$$

where $dt_0 = \beta_0 ds_0$ and $\mathbf{l}_0, \mathbf{n}_0$ are the angular velocity and linear acceleration of the electron measured in (ρ_0, τ_0) at t_0 in (\mathbf{r}, t) .

If

$$\left. \begin{aligned} l_{0\alpha} &= \zeta_{0\alpha b} w_{0b}, \\ n_{0\alpha} &= \zeta_{0\alpha b} f_{0b}, \end{aligned} \right\} . \quad . \quad . \quad (3.24)$$

$\mathbf{w}_0, \mathbf{f}_0$ are the angular velocity and acceleration measured in (\mathbf{R}_0, T_0) .

By eliminating (\mathbf{r}, t) from equations (3.1) and the similar equations for (\mathbf{R}_1, T_1) , (cf. 2.2)

$$\left. \begin{aligned} \mathbf{R}_1 &= \mathbf{R}_0 + \frac{(\beta_0 - 1)}{v_0^2} [\mathbf{R}_0 \times [\mathbf{v}_0 \times d\mathbf{v}_0]] \\ &\quad - \beta_0 T_0 \left\{ d\mathbf{v}_0 + (\beta_0 - 1) \frac{(\mathbf{v}_0 \cdot d\mathbf{v}_0)}{v_0^2} \mathbf{v}_0 \right\}, \\ T_1 &= T_0 - \frac{\beta_0}{c^2} \left\{ \left(\mathbf{R}_0 \cdot \left\{ d\mathbf{v}_0 + (\beta_0 - 1) \frac{(\mathbf{v}_0 \cdot d\mathbf{v}_0)}{v_0^2} \mathbf{v}_0 \right\} \right) \right\} - ds_0, \end{aligned} \right\} \quad (3.3)$$

so that, referred to (\mathbf{R}_0, T_0) , (\mathbf{R}_1, T_1) has a rotation

$$(\beta_0 - 1) [\mathbf{v}_0 \times d\mathbf{v}_0] / v_0^2$$

as well as a velocity

$$\beta_0 d\mathbf{v}_0 + \beta_0 (\beta_0 - 1) (\mathbf{v}_0 \cdot d\mathbf{v}_0) \mathbf{v}_0 / v_0^2.$$

Comparing (3.23) and (3.3), using (3.21), (3.22), (3.24),

$$\mathbf{f}_0 ds_0 = \beta_0 \left\{ d\mathbf{v}_0 + (\beta_0 - 1) \frac{(\mathbf{v}_0 \cdot d\mathbf{v}_0)}{v_0^2} \mathbf{v}_0 \right\}, \quad (3.41)$$

$$\mathbf{w}_0 ds_0 = d\zeta_0 + (\beta_0 - 1) \frac{[\mathbf{v}_0 \times d\mathbf{v}_0]}{v_0^2}, \quad (3.42)$$

If \mathbf{G}_0 is the rate of change of angular velocity referred to (\mathbf{R}_0, T_0) at $t=t_0$ in (\mathbf{r}, t) , by (3.3), since \mathbf{w}_1 is measured in (\mathbf{R}_1, T_1) ,

$$\begin{aligned} \frac{\mathbf{w}_1 - \mathbf{w}_0}{ds_0} &\sim \mathbf{G}_0 + \frac{(\beta_0 - 1)}{v_0^2} \left[\mathbf{w}_0 \times \left[\mathbf{v}_0 \times \frac{d\mathbf{v}_0}{ds_0} \right] \right], \\ \text{i. e. } \mathbf{G}_0 &= \frac{d\mathbf{w}_0}{ds_0} - \frac{(\beta_0 - 1)}{v_0^2} \left[\mathbf{w}_0 \times \left[\mathbf{v}_0 \times \frac{d\mathbf{v}_0}{ds_0} \right] \right]. \quad (3.5) \end{aligned}$$

This means, *inter alia*, that if an electron were to move in a closed path, and its axis of rotation at each moment did not change its direction in the system of coordinates in which the electron was instantaneously at rest, yet after a cycle the direction would be different. In short, \mathbf{G} and $d\mathbf{w}/ds$ are not the same.

For a change of the coordinate system (\mathbf{r}, t) , \mathbf{w} so defined does not change in a simple manner. This difficulty can be got rid of in two ways.

1. If

$$\left. \begin{aligned} (w^1, w^2, w^3) &= \mathbf{w} + (\beta - 1) \frac{(\mathbf{w} \cdot \mathbf{v})}{v^2} \mathbf{v}, \\ w^4 &= \beta \frac{(\mathbf{w} \cdot \mathbf{v})}{c^2}, \end{aligned} \right\} \dots (3.61)$$

then w^μ is a four-vector transformed like dx^μ and having zero time component and space components equal to \mathbf{w} in any system in which the electron is instantaneously at rest.

It therefore always satisfies the relation

$$g_{\mu\nu} w^\mu \frac{dx^\nu}{ds} = 0. \dots (3.62)$$

2. If

$$\left. \begin{aligned} (w_{23}, w_{31}, w_{12}) &= \beta \mathbf{w} + (1 - \beta) \frac{(\mathbf{w} \cdot \mathbf{v})}{v^2} \mathbf{v}, \\ (w_{14}, w_{24}, w_{34}) &= -\beta [\mathbf{v} \times \mathbf{w}], \end{aligned} \right\} \dots (3.71)$$

and $w_{\mu\nu} = -w_{\nu\mu}$, $w_{\mu\nu}$ is an antisymmetrical tensor transformed like $F_{\mu\nu}$ and having zero components (w_{14}, w_{24}, w_{34}) and components (w_{23}, w_{31}, w_{12}) equal to \mathbf{w} in any system in which the electron is instantaneously at rest.

It therefore always satisfies the relations

$$w^{\mu\nu} \frac{dx^\nu}{ds} = 0, \dots (3.72)$$

which are equivalent to three independent relations.

Whether the electron has a definite rotation or not, if \mathbf{w} is a vector defining in a system of coordinates in which its centre is instantaneously at rest, any directed property such as magnetic moment, equations (3.1), (3.3) will apply and (3.5) can be deduced, giving \mathbf{G}_0 , the rate of change of \mathbf{w} referred to a system of coordinates in which the electron is instantaneously at rest in terms of $d\mathbf{w}/ds$. (3.6) and (3.7) will also still hold good as well as (3.41). (3.42), of course, would then have no meaning.

The only part of the sequel for which it is necessary that \mathbf{w} should be an actual angular velocity is § 5, in which the model of a spinning spherical shell of electricity is considered.

4. *A first approximation to the rate of change of direction of the axis of the electron.*

Suppose that, to a first approximation, in the coordinate system (\mathbf{R}, T) in which the electron's centre is instantaneously at rest,

$$\mathbf{f} = -\frac{e}{m}\mathbf{E}', \quad \dots \dots \dots (4.01)$$

$$\mathbf{G} = \lambda[\mathbf{H}' \times \mathbf{w}], \quad \dots \dots \dots (4.02)$$

where \mathbf{E}' , \mathbf{H}' are the field in (\mathbf{R}, T) , $-e$ is the charge, m the mass of the electron, and λ a constant not yet determined.

Then it follows by (3.5), (3.41), (2.3), expressing $d^2\mathbf{r}/ds^2$, d^2t/ds^2 , $d\mathbf{w}/ds$ in terms of $d\mathbf{r}/ds$, dt/ds , \mathbf{w} , \mathbf{E} , \mathbf{H} , that, to a first approximation,

$$\left. \begin{aligned} \frac{d^2\mathbf{r}}{ds^2} &= -\frac{e}{m} \left\{ \frac{dt}{ds} \mathbf{E} + \frac{1}{c} \left[\frac{d\mathbf{r}}{ds} \times \mathbf{H} \right] \right\}, \\ \frac{d^2t}{ds^2} &= -\frac{e}{mc^2} \left(\frac{d\mathbf{r}}{ds} \cdot \mathbf{E} \right), \end{aligned} \right\} \dots \dots (4.11)$$

which are, of course, equivalent to (1.72), remembering that the electronic charge is $-e$, and

$$\begin{aligned} \frac{d\mathbf{w}}{ds} &= \left[\left\{ \left\{ \frac{e}{mc} + \beta \left(\lambda - \frac{e}{mc} \right) \right\} \mathbf{H} + \frac{(1-\beta)}{v^2} \left(\lambda - \frac{e}{mc} \right) (\mathbf{H} \cdot \mathbf{v}) \mathbf{v} \right. \right. \\ &\quad \left. \left. + \left(\frac{e}{mc^2} \frac{\beta^2}{1+\beta} - \frac{\lambda\beta}{c} \right) [\mathbf{v} \times \mathbf{E}] \right\} \times \mathbf{w} \right]. \end{aligned} \quad (4.121)$$

This last is considerably simpler when $\lambda = \frac{e}{mc}$, when it takes the form

$$\frac{d\mathbf{w}}{ds} = \left[\left\{ \frac{e}{mc} \mathbf{H} - \frac{e}{mc^2} \frac{\beta}{1+\beta} [\mathbf{v} \times \mathbf{E}] \right\} \times \mathbf{w} \right]. \quad (4.122)$$

In this case, to the same approximation,

$$\frac{dw^\mu}{ds} = \frac{e}{mc} F^\mu{}_\nu w^\nu, \quad \dots \dots \dots (4.123)$$

and

$$\frac{dw_{\mu\nu}}{ds} = \frac{e}{mc} \{ F^\sigma{}_\nu w_{\sigma\mu} - F^\sigma{}_\mu w_{\sigma\nu} \}. \quad \dots \dots (4.124)$$

The more complicated forms when $\lambda \neq e/mc$ involving \mathbf{v} explicitly on the right-hand side can be found easily if required.

The first approximation equation (4.124) including the coefficient e/mc can also be deduced from the following assumptions.

To a first approximation :—

1. The equations of motion of the centre of the electron are (1.73)

$$\frac{d^2 x^\mu}{ds^2} = \frac{e}{mc} F^\mu{}_\nu \frac{dx^\nu}{ds}.$$

2. The antisymmetrical tensor $w_{\mu\nu}$ has its rate of change given by equations that are invariant for Lorentz transformations.

3. $dw_{\mu\nu}/ds$ is a homogeneous linear function of the field ($F_{\mu\nu}$).

4. $dw_{\mu\nu}/ds$ contains no terms higher than quadratic functions of $w_{\mu\nu}$, dx^μ/ds .

5. The relation $w_{\mu\nu} dx^\nu/ds = 0$ is preserved

$$\left(\text{i. e. } w_{\mu\nu} \frac{d^2 x^\nu}{ds^2} + \frac{dw_{\mu\nu}}{ds} \frac{dx^\nu}{ds} = 0 \right).$$

6. The relation $w_{\mu\nu} w_{\sigma\tau} g^{\mu\sigma} g^{\nu\tau} = \text{const.}$ is preserved

$$\left(\text{i. e. } w_{\sigma\tau} g^{\mu\sigma} g^{\nu\tau} \frac{dw_{\mu\nu}}{ds} = 0 \right).$$

For assumptions 2, 3, and 4 only allow an equation of the form

$$\frac{dw_{\mu\nu}}{ds} = K_{\mu\nu} - K_{\nu\mu},$$

where

$$\begin{aligned} K_{\mu\nu} = & A F^\sigma{}_\nu w_{\sigma\mu} + B \Phi^\sigma{}_\nu w_{\sigma\mu} + C F_{\mu\nu} + D \Phi_{\mu\nu} \\ & + E F_{\sigma\nu} \frac{dx^\sigma}{ds} \frac{dx^\tau}{ds} g_{\mu\tau} + F \Phi_{\sigma\nu} \frac{dx^\sigma}{ds} \frac{dx^\tau}{ds} g_{\mu\tau}, \end{aligned}$$

and A, B, C, D, E, F are constants.

Assumptions 5 and 1 then give $A = e/mc$, $B = 0$, $C = 0$, $D = 0$, and assumption 6 gives $E = 0$, $F = 0$.

Similar assumptions for w_μ lead to a similar result.

The next stage would naturally be to seek the second approximation terms in equations (4.11) of motion of the centre of the electron. There does not, however, appear to be any definite way of finding them as they depend on the assumptions made as to the constitution of the electron. As will appear below, to obtain the observed Zeeman effect

and first order doublet separation it seems only to be necessary that they should be such that there is some "angular momentum" which is conserved when the electron revolves in a central field.

Hamilton's principle does not seem to be applicable as the kinematical conditions do not seem to have any meaning for a variation to an adjacent motion not necessarily kinematically possible. Equations in a Lagrange form with extra terms to preserve the kinematical conditions of the usual form can be obtained. The test of their truth must be whether they can give higher order terms in the doublet separation in agreement with experimental results, which, as is well known, are given to a high approximation by Sommerfeld's formula. The equations can also be put in Appell's form*. Perhaps the assumption that the motion of the electron can be described in terms of its position at any time and a single further vector \mathbf{w} is too restrictive.

5. *The Abraham spinning electron.*

It is interesting to see what some particular model of a spinning electron gives for its equations of motion. The simplest is that of Abraham†, a spinning sphere of electricity. As Uhlenbeck and Goudsmit pointed out, he showed that for a uniform spherical shell the coefficient λ in equation (4.02) is e/mc .

Assume:—

1. The electron is a distribution of charge which, in a Galilean system in which its centre is instantaneously at rest is instantaneously spherically symmetrical and rotating like a rigid body with angular velocity \mathbf{w} .

2. In such a system the total couple and total force on the electron due to the external field and to its own field, given by retarded potentials, are zero.

These assumptions differ from those of Abraham only in that they imply a Lorentz contraction of the electron, while he supposed it to remain spherical in a system fixed in the "æther."

It follows, agreeing with Abraham, that for $\mathbf{v} = 0$, approximately

$$m\mathbf{f} = -e\mathbf{E} + l\nabla(\mathbf{w} \cdot \mathbf{H}), \quad \left. \begin{array}{l} \\ \end{array} \right\} \dots (5.1)$$

$$I \frac{d\mathbf{w}}{dt} = cl [\nabla \times \mathbf{E}] + l [\mathbf{w} \times \mathbf{H}], \quad \left. \begin{array}{l} \\ \end{array} \right\} \dots (5.2)$$

* M. P. Appell, *Comptes Rendus*, 1911, p. 1197.

† M. Abraham, *loc. cit.*

where $-e$ is the total charge, lw the magnetic moment, and

$$m = \frac{2}{3c^2} \iint \frac{d\rho_1 d\rho_2}{D},$$

$$I = \frac{1}{2e^2} \iint \left\{ \frac{2(\mathbf{P}_1 \cdot \mathbf{P}_2)}{D} + \frac{1}{3} \frac{[\mathbf{P}_1 \times \mathbf{P}_2]^2}{D^3} \right\} d\rho_1 d\rho_2,$$

$$l = \frac{1}{3c} \int \mathbf{P}^2 d\rho.$$

The integrations being over the electron, $d\rho_1, d\rho_2$ elements of charge at $\mathbf{P}_1, \mathbf{P}_2$ from the centre, while $D^2 = (\mathbf{P}_1 - \mathbf{P}_2)^2$.

For a spherical shell radius a ,

$$m = \frac{2}{3} \frac{e^2}{ac^2}, \quad I = \frac{2}{9} \frac{ae^2}{c^2}, \quad l = \frac{a^2e}{3c}.$$

The terms neglected in (5.1) and (5.2) are terms in $d\mathbf{f}/dt, d^2\mathbf{f}/dt^2$ etc., $d^2\mathbf{w}/dt^2, d^3\mathbf{w}/dt^3$ etc., and squares, products, and higher powers of \mathbf{f}, \mathbf{w} and their derivatives, as well as terms in the second and higher differential coefficients of the external field.

It is to be noted that if the terms $l\nabla(\mathbf{w} \cdot \mathbf{H}), cl[\nabla \times \mathbf{E}] = -l\dot{\mathbf{H}}$, depending on the rates of change of the field are left out, the equations (4.01), (4.02) considered above are obtained with $\lambda = e/mc$.

Further, the term $l\nabla(\mathbf{w} \cdot \mathbf{H})$ is not that which would be expected to be the force on a small magnet in a non-uniform field. The latter is $l(\mathbf{w} \cdot \nabla)\mathbf{H}$. The difference between these expressions

$$\begin{aligned} l(\mathbf{w} \cdot \nabla)\mathbf{H} - l\nabla(\mathbf{w} \cdot \mathbf{H}) &= -l[\mathbf{w} \times [\nabla \times \mathbf{H}]] \\ &= -(l/c)[\mathbf{w} \times \dot{\mathbf{E}}], \end{aligned}$$

while zero if the electric field in the system of coordinates in which the electron is instantaneously at rest is not changing, is not usually zero.

This model has disadvantages that seem to deprive it of any physical meaning.

(1) If $Iw = h/4\pi$ as will be required,

$$\begin{aligned} |a\mathbf{w}| &= (9c^2/4e^2) |I\mathbf{w}| \\ &\approx 200c, \end{aligned}$$

which seems absurd.

(2) The term in fw^2 was neglected in equation (5.1). This term would represent a magnetic contribution to the mass, and since there would be no corresponding contribution to the angular momentum, the ratio $\lambda = e/mc$ would be disturbed. Further, it has been assumed that while internal forces constrain the electron so that it always has spherical shape instantaneously in a system in which its centre is at rest, these forces do not contribute to the total force and couple in that coordinate system. This will not in general be true for other coordinate systems.

Since Webster has shown that internal forces can be assumed which will cancel the magnetic mass*, the second objection loses some of its force. Moreover, a rapidly spinning electron would not be expected to remain spherical. This model is, however, interesting as showing that there is nothing inherently impossible about the ratio e/mc .

I think we must look towards the general relativity theory for an adequate solution of the problem of the "structure of the electron"; if indeed this phrase has any meaning at all and if it can be possible to do more than to say how an electron behaves in an external field.

6. *The secular change in the direction of the axis of an electron revolving in an orbit.*

It makes no difference which of \mathbf{w} , w^μ , $w_{\mu\nu}$ is used to find the secular change. If the electron revolves under a central electric field Ze/r^2 and a constant external magnetic field \mathbf{H} , from (4.122) approximately

$$\frac{d\mathbf{w}}{dt} = \left[\left\{ \frac{e}{mc} \mathbf{H} - \frac{e^2}{mc^2} \frac{1}{2} \left[\mathbf{v} \times \frac{\mathbf{r}Z}{r^3} \right] \right\} \times \mathbf{w} \right] \dots (6.1)$$

$\mathbf{K} = m[\mathbf{r} \times \mathbf{v}]$ is the angular momentum of revolution, and can be supposed nearly constant during one revolution, so the secular change in the direction of the spin axis is given by rotation

$$\frac{e}{mc} \mathbf{H} + \frac{1}{2} \frac{e^2}{mc^2} \frac{\bar{Z}}{r^3} \mathbf{K}, \dots \dots (6.2)$$

where \bar{Z}/r^3 is a time mean over a revolution. The occurrence of the factor "1/2" in the second part of (6.1) is due to the difference between \mathbf{G} and $d\mathbf{w}/ds$ caused by the rotation between (\mathbf{R}_0, T_0) and (\mathbf{R}_1, T_1) . If the expression for \mathbf{G} were taken instead of (6.1) to be $d\mathbf{w}/ds$, as might be done in the

* D. L. Webster, Phys. Rev. ix. p. 484 (1917).

hope that such "relativity rotation" would prove small, then the secular rotation would seem to be

$$\frac{e}{mc} \mathbf{H} + \frac{e^2}{mc^2} \frac{\bar{Z}}{r^3} \mathbf{K}, \dots \dots \dots (6.3)$$

or if $\lambda = e/2mc$,

$$\frac{e}{2mc} \mathbf{H} + \frac{e^2}{2mc^2} \frac{\bar{Z}}{r^3} \mathbf{K}. \dots \dots \dots (6.4)$$

It is to be noted that, for a Coulomb field, (Z independent of r),

$$\frac{1}{2} \frac{e^2}{mc^2} \frac{\bar{Z}}{r^3} \mathbf{K} = \sigma$$

is equal to the rate of precession of the perihelion of the orbit in its own plane due to the Sommerfeld relativity effect.

The secular change in the orbit will be the relativity plus screening precession in its own plane, the Larmor precession of its normal about the external magnetic field, and the effect of the unknown second order terms in equations (4.11). If there is to be any "angular momentum" that is conserved at all and the effect is not zero, this effect can hardly be other than some small deformation of the orbit and a rotation of the normal proportional to ω . That is, the change in direction of the orbit normal will be a rotation of the form

$$\frac{e}{2mc} \mathbf{H} + \omega \mathbf{w}, \dots \dots \dots (6.5)$$

where ω depends on the shape and size of the orbit only.

In the absence of external field the equations giving the secular change in plane of the orbit and in \mathbf{w} would then be

$$\left. \begin{aligned} \frac{d\mathbf{w}}{dt} &= \frac{\sigma}{K} [\mathbf{K} \times \mathbf{w}], \\ \frac{d\mathbf{K}}{dt} &= \omega [\mathbf{w} \times \mathbf{K}], \end{aligned} \right\}$$

so that

$$\frac{d}{dt} \left(\mathbf{K} + \frac{K}{\sigma} \omega \mathbf{w} \right) = 0.$$

$\mathbf{K} + (K/\sigma) \omega \mathbf{w}$ is what is secularly unchanged. If and only if $(K/\sigma) \omega$ is the same for all orbits, can this be divided into the angular momentum of the orbit and an angular momentum independent of the orbit to be attributed to the electron.

If $(K/\sigma)\omega = \Omega$, it would be natural to call Ω the angular momentum of the electron, (6.5) becomes

$$\frac{e}{2mc} \mathbf{H} + \frac{\sigma}{K} \Omega, \quad (6.6)$$

and the equations to determine secular changes of the orbital plane and electronic axis are

$$\frac{d\Omega}{dt} = \left[\left(\frac{e}{mc} \mathbf{H} + \frac{\sigma}{K} \mathbf{K} \right) \times \Omega \right], \quad \left. \vphantom{\frac{d\Omega}{dt}} \right\} . . (6.71)$$

$$\frac{d\mathbf{K}}{dt} = \left[\left(\frac{e}{2mc} \mathbf{H} + \frac{\sigma}{K} \Omega \right) \times \mathbf{K} \right] \quad \left. \vphantom{\frac{d\mathbf{K}}{dt}} \right\} . . (6.72)$$

It is to be noted that using the equation (5.1) obtained from the model of the Abraham electron and averaging just gives (6.72) with $\Omega = I\omega$.

If $m\mathbf{f}_0 = -e\mathbf{E}_0 + (le/mc)(\omega \cdot \nabla)\mathbf{H}_0$ is used instead of (5.1),

$$\frac{d\mathbf{K}}{dt} = \left[\left(\frac{e}{2mc} \mathbf{H} + \frac{\sigma}{K} 2I\omega \right) \times \mathbf{K} \right]$$

would be obtained instead of (6.72). However, as $\mathbf{K} + 2I\omega$ would then, in the absence of external field, be conserved, it would be natural to put $2I\omega = \Omega$ and still obtain (6.71), (6.72).

The form of these equations depends only on that of (6.2) and on there being an "angular momentum" secularly conserved in the absence of external magnetic field of which part is the ordinary angular momentum of revolution and part, constant in magnitude, can be assigned to the electron.

7. The application of the correspondence principle to obtain approximate term values.

In order to compare with observation, the term values that would be obtained for an electron moving in central electric field Ze/r^2 under a small external magnetic field \mathbf{H} will be obtained.

The first approximation will be the Keplerian ellipse, the term values being

$$-\frac{RZ^2}{n^2},$$

where R is the Rydberg number, n the total quantum number.

Next the orbit precesses in its own plane with the Sommerfeld relativity precession giving the term

$$Z^4 R x^2 \left(\frac{3}{4} \frac{1}{n^4} - \frac{1}{n^3 k} \right),$$

where

$$x = 2\pi e^2 / ch = 7.29.10^{-3} \quad \text{and} \quad R x^2 Z^4 / n^3 k^2 = \sigma / 2\pi c,$$

while the normal to the orbit and the spin axis precess with angular velocity

$$\frac{\sigma}{K} \mathbf{J}, \quad \text{where } \mathbf{J} = \mathbf{K} + \mathbf{\Omega}$$

about the direction \mathbf{J} , while \mathbf{J} precesses about \mathbf{H} with angular velocity

$$\frac{e}{mc} \mathbf{H} \frac{(\mathbf{\Omega} \cdot \mathbf{J})}{J} + \frac{e}{2mc} \mathbf{H} \frac{(\mathbf{K} \cdot \mathbf{J})}{J},$$

since for small \mathbf{H} , (6.71), (6.72) give first

$$\left. \begin{aligned} \frac{d\mathbf{\Omega}}{dt} &= \frac{\sigma}{K} [\mathbf{J} \times \mathbf{\Omega}], \\ \frac{d\mathbf{K}}{dt} &= \frac{\sigma}{K} [\mathbf{J} \times \mathbf{K}], \end{aligned} \right\}$$

and then, averaging over the first rotation,

$$\frac{d\mathbf{J}}{dt} = \frac{e}{mc} [\mathbf{H} \times \mathbf{J}] \frac{(\mathbf{\Omega} \cdot \mathbf{J})}{J^2} + \frac{e}{2mc} [\mathbf{H} \times \mathbf{K}] \frac{(\mathbf{K} \cdot \mathbf{J})}{J^2}.$$

Whether \mathbf{H} is small or not equations (6.71), (6.72) are just those which would be obtained if the system were Hamiltonian with perturbation function, in addition to the Sommerfeld relativity correction,

$$F = \frac{\sigma}{K} (\mathbf{K} \cdot \mathbf{\Omega}) + \frac{e}{mc} (\mathbf{H} \cdot \mathbf{\Omega}) + \frac{e}{2mc} (\mathbf{H} \cdot \mathbf{K}).$$

Thus in any case, term values \bar{F}/ch will give the line frequency differences that correspond to equations (6.71), (6.72), where \bar{F} is the mean value of F over the motion.

For \mathbf{H} small, writing $K = kh/2\pi$, $J = jh/2\pi$, $\mathbf{\Omega} = rh/2\pi$, $(\mathbf{J} \cdot \mathbf{H})/H = mh/2\pi$, and expressing the cosines in terms of k , j , r , and m , the term value is

$$\frac{1}{ch} \left\{ \frac{j^2 - k^2 - r^2}{2kr} \frac{rh}{2\pi} \sigma + \left(\frac{eH}{2mc} \right) \frac{mh}{2\pi} \left\{ \frac{k^2 + j^2 - r^2}{2j^2} + 2 \frac{(r^2 + j^2 - k^2)}{2j^2} \right\} \right\},$$

i. e.

$$Z^4 R x^2 \left\{ \frac{j^2 - k^2 - r^2}{2n^3 k^3} \right\} + m \frac{O_{\mathbf{H}}}{c} \left\{ \frac{k^2 + j^2 - r^2}{2j^2} + 2 \frac{(r^2 + j^2 - k^2)}{2j^2} \right\}.$$

This gives, collecting all the terms, term value

$$-\frac{RZ^2}{n^2} + Z^4Rx^2\left(\frac{3}{4}\frac{1}{n^4} - \frac{1}{n^3k}\right) + Z^4Rx^2\left(\frac{j^2 - k^2 - r^2}{2n^3k^3}\right) \\ + m\frac{O_H}{c}\left\{\frac{k^2 + j^2 - r^2}{2j^2} + 2\frac{(r^2 + j^2 - k^2)}{2j^2}\right\}. \quad (7.1)$$

For $\Omega = h/4\pi$, $r = \frac{1}{2}$.

Starting from the treatment of the motion of a particle in a Coulomb field due independently to Pauli and Dirac, Heisenberg and Jordan (*loc. cit.*) have shown that on the new theory the above formula must be replaced by

$$-\frac{RZ^2}{n^2} + Z^4Rx^2\left(\frac{3}{4}\frac{1}{n^4} - \frac{1}{n^3(k + \frac{1}{2})}\right) \\ + Z^4Rx^2\left(\frac{j(j+1) - k(k+1) - r(r+1)}{2n^3k(k + \frac{1}{2})(k+1)}\right) \\ + m\frac{O_H}{c}\left\{\frac{k(k+1) + j(j+1) - r(r+1)}{2j(j+1)} + 2\frac{(r(r+1) + j(j+1) - k(k+1))}{2j(j+1)}\right\}, \quad (7.2)$$

where $r = \frac{1}{2}$

$$n = 1, 2, 3 \dots$$

$$k = 0, 1, \dots (n-1)$$

$$j = k + \frac{1}{2} \text{ or } k - \frac{1}{2} \text{ unless } k = 0, \text{ when } j = \frac{1}{2}$$

$$m = -j, -j+1, \dots, 0, \dots, j-1, j,$$

except that they have not proved that the third term

$$Z^4Rx^2\left(\frac{j(j+1) - k(k+1) - r(r+1)}{2n^3k(k + \frac{1}{2})(k+1)}\right)$$

takes the form it should for S-terms, i. e. $k=0$, $j = \frac{1}{2} = r$, viz.

$$Z^4Rx^2\frac{1}{2n^3(k + \frac{1}{2})} \dots \dots \dots (7.3)$$

The last term of (7.2) gives the anomalous Zeeman effect correctly.

If the form (7.3) be granted for the S-term, the second and third terms give the "relativity" fine structure correctly according to Uhlenbeck and Goudsmit's scheme with the

observed doublet separation. In fact, if $Z=1$ these two terms reduce to

$$Rx^2 \left\{ \frac{3}{4} \frac{1}{n^4} - \frac{1}{n^3(k+1)} \right\} \quad \text{for } j = k + \frac{1}{2},$$

$$Rx^2 \left\{ \frac{3}{4} \frac{1}{n^4} - \frac{1}{n^3k} \right\} \quad \text{for } j = k - \frac{1}{2}, \quad k \neq 0.$$

In general the coefficients Z of the different terms in (6.2) will be affected by different screening constants as they arise as averages of different expressions. In any case the doublet separation, the difference of the term values for the same n , k , and different j is

$$Z^4 Rx^2 / n^3 k (k + 1).$$

It is to be noted that if instead of the correct expression (6.2), involving the kinematical rotation of the axis discussed above, expression (6.3) had been used, the anomalous Zeeman effect would have been obtained correctly, but twice the observed doublet separation; if expression (6.4), the correct doublet separation but a normal Zeeman effect.

Finally the value $\Omega = h/4\pi$ is necessary for doublets to be obtained (as against triplets, etc., Heisenberg, *loc. cit.*).

8. *A summary of the logical train by which the Goudsmit-Uhlenbeck theory connects the anomalous Zeeman effect with the optical and relativity doublets and accounts for them both as manifestations of the magnetic properties of the electron.*

Starting from the idea that the anomalous Zeeman effect may be explained by the electron itself having a magnetic moment, it is seen that Landé g -values different from unity can be obtained by supposing that the axis of the electron precesses about a magnetic field with angular velocity other than the Larmor rotation. In fact angular velocity e/mc , twice the Larmor precession, must be assumed. If it be assumed that it has such a precession about the magnetic field in a system of axes in which its centre is instantaneously at rest, the secular rate of change of direction of its axis when it revolves in an orbit can be found. Assuming that there is some total angular momentum that is secularly conserved, that which the electron itself adds to that of the orbit having magnitude $h/4\pi$, the secular motion of the system can be found. An approximate formula for the doublet and Zeeman effect separations follows. The new quantum mechanics of Heisenberg transforms this into a formula

which fits the observed doublet separations and Zeeman effect exactly, as far as first order terms in the relativity correction are sufficient.

In conclusion, I wish to express my appreciation of the encouragement and help of Professor Bohr and Dr. Kramers. My thanks are due to Dr. Pauli and Dr. Heisenberg for their helpful criticism.

Universitetets Institut for Teoretisk Fysik,
Copenhagen.

Note added later.

Since the above was written Frenkel has published a paper* in which he shows how the consistency of the Zeeman effect and multiplet structure can be obtained from a similar Lagrange function with the same kinematical conditions. (3.72, 1.02.)

Perhaps I may be allowed to state a little more fully than above the position as it appears to me. The difficulty is that if we use Hamilton's principle we must define or take as defined varied motions in which the kinematical conditions are not satisfied, and the equations cannot be interpreted in the above way as representing a rotation in the ordinary sense. Equations for the variations of the "velocities corresponding to quasi-coordinates" $w_{\mu\nu}$ or their equivalent must be assumed in the form

$$\delta dw_{\mu\nu} = d\delta w_{\mu\nu} + g^{\sigma\tau}(\delta w_{\mu\sigma} dw_{\tau\nu} - dw_{\mu\sigma} \delta w_{\tau\nu}), \quad (9.2)$$

where $dw_{\mu\nu}$ means $w_{\mu\nu} ds$, and $\delta w_{\mu\nu}$ are the independent variations, if the analogy with ordinary mechanics is to remain. "True coordinates" giving rise to (9.2) can be found as a kind of formal extension of Eulerian angles (*e.g.* the equations (9.5) below with $d\zeta_{\mu\nu}$ replaced by $w_{\mu\nu} ds$ would lead to (9.2)), but there seems to be no physical interpretation of them. I have tried to use as Lagrange function, originally obtained in connexion with the Abraham electron model,

$$L = -\frac{1}{2}mg_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} + \frac{e}{c} k_\mu \frac{dx^\mu}{ds} + \frac{1}{4} I w_{\mu\nu} w_{\sigma\tau} g^{\mu\sigma} g^{\nu\tau} - \frac{\lambda}{2} w_{\mu\nu} F^{\mu\nu}. \quad (9.3)$$

If $\lambda = \frac{Ie}{mc}$ the kinematical conditions are automatically satisfied to the first order—as the argument given above

* J. Frenkel, *Zeit. für Physik*, xxxvii, p. 243.

shows. As we have already contemplated motions not satisfying the kinematical conditions, it is natural to see what happens if they are discarded altogether. The equations then take the wonted Hamiltonian form, and the usual methods of solution can be applied safely at any rate to the first order.

Frenkel uses a Lagrange function equivalent to

$$L = \frac{e}{c} k_{\mu} \frac{dx^{\mu}}{ds} + \frac{1}{4} I w_{\mu\nu} w_{\sigma\tau} g^{\mu\sigma} g^{\nu\tau} + \frac{\lambda}{2} w_{\mu\nu} F^{\mu\nu}.$$

The change of sign in the last term may be brought about by denoting by $w_{\mu\nu}$ what I have called $w_{\nu\mu}$, or by $-e$ what I have called e . Use of the kinematical conditions, which is now necessary if nonsense is to be avoided, leads him to first order secular equations identical with those obtained in § 6 of this paper.

The following results more general than those in § 3 which I obtained at the same time may be of interest.

We can suppose the electron's configuration to be determined by giving the position of its centre, x^{μ}_0 , and the orientation of a Galilean coordinate system ξ^{μ} with origin at its centre, which can be described as in (3.1), (3.21) in terms of $\theta, \phi, \psi, \mathbf{v}$; $\beta = (1 - \mathbf{v}^2/c^2)^{-\frac{1}{2}}$, where \mathbf{v} is not now necessarily the velocity of the centre. Thus

$$\xi^{\mu} = A^{\mu}_{\nu} (x^{\nu} - x^{\nu}_0),$$

where A^{μ}_{ν} depend on $\theta, \phi, \psi, \mathbf{v}$. After interval ds ,

$$\xi^{\mu'} = A^{\mu}_{\nu'} (x^{\nu} - x^{\nu}_0') = A^{\mu}_{\nu'} y^{\nu}.$$

Defining $y^{\mu'}$ by

$$\xi^{\mu'} = A^{\mu}_{\nu} y^{\nu'},$$

it follows that

$$y^{\mu'} = d\zeta^{\mu}_{\nu} y^{\nu} + y^{\mu},$$

where $d\zeta^{\mu}_{\nu}$ may, without prejudice, be called the "rotation" of the electron. It is easily shown that

$$\delta d\zeta^{\mu}_{\nu} = d\delta\zeta^{\mu}_{\nu} + \delta\zeta^{\mu}_{\sigma} d\zeta^{\sigma}_{\nu} - d\zeta^{\mu}_{\sigma} \delta\zeta^{\sigma}_{\nu} \quad \dots \quad (9.4)$$

if $\theta, \phi, \psi, \mathbf{v}$ be regarded as (six) "true coordinates" capable of variation independently of x^{μ}_0 , while

$$(d\zeta_{23}, d\zeta_{31}, d\zeta_{22}) = \beta d\zeta + \frac{(1-\beta)}{\mathbf{v}^2} \mathbf{v}(d\zeta \cdot \mathbf{v}) + \frac{(1-\beta)}{\mathbf{v}^2} [\mathbf{v} \times d\mathbf{v}],$$

$$(d\zeta_{14}, d\zeta_{24}, d\zeta_{34}) = -\beta d\mathbf{v} - \beta [\mathbf{v} \times d\zeta] + \frac{\beta(1-\beta)}{\mathbf{v}^2} \mathbf{v}(\mathbf{v} \cdot d\mathbf{v}), \quad \dots \quad (9.5)$$

where $d\zeta$ is given by (3.22).

The condition that if initially

$$(\beta_{\nu}, \beta) = \frac{dx^{\nu}}{ds}$$

this shall continue to hold is

$$\frac{dx^{\nu}}{ds} \frac{d\xi_{\nu}^{\mu}}{ds} = -\frac{d^2x^{\nu}}{ds^2}, \dots \dots \dots (9.6)$$

which are four independent conditions replacing (3.72), (1.02).

If these hold the equations reduce to those of § 3, with

$$\frac{d\xi^{\mu\nu}}{ds} = w^{\mu\nu} + \frac{1}{c^2} \left\{ \frac{dx^{\nu}}{ds} \frac{d^2x^{\mu}}{ds^2} - \frac{dx^{\mu}}{ds} \frac{d^2x^{\nu}}{ds^2} \right\}. \dots (9.7)$$

If this picture is adopted, $d\xi_{\mu\nu}$, with (9.6) as kinematical conditions, would seem to replace $w_{\mu\nu}ds$ as the fundamental rotation tensor, as the former retains some physical meaning when (9.6) does not hold (in which case $w_{\mu\nu}$ might perhaps be *defined* by (9.7)) and on this picture $d\xi_{\mu\nu}$ satisfies (9.4), while $w_{\mu\nu}$ does not in general satisfy (9.2). Equations (9.6) do not seem to be suitable to be kinematical conditions connected with a Lagrange function.

Taking a Lagrange function of the form (9.3), $d\xi_{\mu\nu}/ds$ replacing $w_{\mu\nu}$,

$$\begin{aligned} L = & -\frac{1}{2}mg_{\mu\nu} \frac{dx^{\mu}}{ds} \frac{dx^{\nu}}{ds} + \frac{e}{c}k_{\mu} \frac{dx^{\mu}}{ds} \\ & + \frac{1}{4}I \frac{d\xi_{\mu\tau}}{ds} \frac{d\xi_{\sigma\nu}}{ds} g^{\mu\sigma} g^{\nu\tau} - \frac{1}{2} \frac{Ie}{mc} \frac{d\xi_{\mu\nu}}{ds} F^{\mu\nu}, \end{aligned} \quad (9.8)$$

equations (9.6) are approximately satisfied better than (3.72) (1.02) with (9.3). To the first order this Lagrange function and the corresponding Hamiltonian function seem to be sufficient.