obtained no trace of a line at 7.8 kev using a proportional counter with *glass* walls and a thin mica window, and he suggests that the 7.8-kev line, found by the above-mentioned authors, has its origin in the fluorescent x-rays of copper (8.05 kev) excited in the wall of the counter by the L x-rays of RaD. We have since confirmed by absorption experiments, using the same *brass* walled counter with which we had previously observed the "7.8-kev line," that this line is in fact produced by fluorescent x-rays from the copper and zinc in the counter wall.

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APPENDIX

Preparation of Radioactive Lead Tetramethyl

RaD (dissolved in nitric acid and in equilibrium with its products) was added to a carrier solution containing usually from about 5–10g of ordinary lead in the form of lead nitrate. By the addition of excess dilute HCl the RaD was precipitated with carrier as lead chloride, and this precipitate was washed several times with water and acetone and then thoroughly dried. It was found by preliminary experiment that the lead chloride so obtained contained less than 3 percent of the RaE and RaF originally in equilibrium with the RaD. This precipitate was refluxed with an ethereal solution of the freshly prepared Grignard reagent (methyl magnesium iodide) for about 3 hours. After destroying any excess Grignard by the addition of water, the resulting ethereal solution of lead tetramethyl was separated, dried with calcium chloride, and distilled, the lead tetramethyl (bp 110°C) being collected.

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Proof That Probability Density Approaches $|\psi|^2$ in Causal Interpretation of the Quantum Theory

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In two previous papers a causal interpretation of the quantum theory was developed which involved the hypothesis that a quantum-mechanical system contains a precisely defined particle variable $\mathbf{x}$ but that, at present, we are restricted to calculating the probability density $P(\mathbf{x}, t)$ that the particle is at the position $\mathbf{x}$. It was shown that the assumption that $P(\mathbf{x}, t)=|\psi(\mathbf{x}, t)|^2$ is consistent, in the sense that if it holds initially, the equations of motion of the particles will cause this relation to be maintained for all time. In this paper, we extend the theory by showing that as a result of random collisions, an arbitrary probability density will ultimately decay into one with a density of $|\psi(\mathbf{x}, t)|^2$. Since all quantum-mechanical experiments to date have been concerned with statistical ensembles of systems that have been colliding with other systems for a very long time, it is therefore inevitable that as we draw samples from such ensembles, the probability density of systems with particles at the point $\mathbf{x}$ will be equal to $|\psi(\mathbf{x}, t)|^2$.

In the previous papers we also pointed out that, within the conceptual framework of the causal interpretation, it was possible to suggest mathematical theories more general than are permitted by the usual interpretation and that these more general theories might be needed in the domain of 10^{-13} cm, where present theories seem to fail. However, if these more general theories should apply at the level of 10^{-13} cm, then there would be a tendency to create discrepancies between P and $|\psi|^2$, a tendency whose cumulative effects should be felt even at the atomic level, where the more general theory ought to approach the usual theory. However, because those discrepancies have been shown to die out as a result of collisions, we can expect that under normal conditions the difference between P and $|\psi|^2$ would be negligible. Conditions are suggested, however, in which this difference might be appreciable, and experiments are indicated which might be able to test for the existence of such discrepancies.

I. INTRODUCTION

IN two recent papers,[1] (to be denoted hereafter by I and II, respectively) the author has proposed a causal reinterpretation of the quantum theory, based on the following hypotheses:

(a) A quantum-mechanical system, such as an electron, consists basically of a particle having a precisely defined position, which varies continuously as a function of the time.

(b) This particle is acted on not only by the classical potential $V(\mathbf{x}, t)$ but also by an additional quantum-potential $U(\mathbf{x}, t)$, which is important at the atomic level but negligible at the macroscopic level.

(c) If we write $\psi=Re^{iS/\hbar}$, where ψ is the wave function and R and S are real, then the quantum potential is given by

$$U(\mathbf{x}, t)=-(\hbar^2/2m)\nabla^2R(\mathbf{x}, t)/R(\mathbf{x}, t). \quad (1)$$

The equation of motion of the particle then takes the form

$$md^2\mathbf{x}/dt^2=-\nabla\{U(\mathbf{x}, t)+V(\mathbf{x}, t)\}. \quad (2)$$

To obtain the same predictions for all experimental results as are obtained from the usual interpretation of the quantum theory it is necessary, however, to make the following additional special assumptions (see paper I, p. 171):

(1) The ψ field satisfies Schrödinger's equation.

(2) The particle velocity is restricted to $\mathbf{v}=\nabla S(\mathbf{x})/m$.

[1] D. Bohm, Phys. Rev. **85**, 166 (1952) (paper I); **85**, 180 (1952) (paper II). See also, Phys. Rev. **87**, 389 (1952).

(3) We do not predict or control the precise location of a particle, but have a statistical ensemble of particles with a probability density, $P(\mathbf{x})=|\psi(\mathbf{x})|^2$.

It was suggested in paper I, Sec. 4, that if we gave up the special assumptions listed above, the causal interpretation of the quantum theory would permit the construction of new types of theories which might be needed in the treatment of certain domains, such as that involving distances of the order of 10^{-13} cm, where existing theories do not seem to be adequate. In support of this suggestion it was actually demonstrated in paper I, Sec. 9, that theories could be formulated, in which assumptions (1) and (2) ceased to be valid at the level of 10^{-13} cm but became approximately valid at the atomic level, where the usual theory is known to be applicable. No similar proof was given in papers I and II, however, with regard to assumption (3), although the conjecture was made in paper II, Sec. 7, that the effects of collisions and other random processes would be to cause any differences between $P(\mathbf{x})$ and $|\psi(\mathbf{x})|^2$ to decay with the passage of time, and thus to tend to establish the validity of assumption (3) at the atomic level, even under conditions in which it failed in the domain of 10^{-13} cm. The object of the present paper is to prove the above conjecture.

We begin our proof by noting that, in general, the probability density $P(\mathbf{x})$ may be an arbitrary function, restricted only by the requirement that it satisfies the conservation equation

$$(\partial P/\partial t)+\operatorname{div}(P\mathbf{v})=0. \tag{3}$$

But as shown in paper I, Sec. 4, provided that ψ is a solution of Schrödinger's equation and that $\mathbf{v}=\nabla S/m$, (i.e., provided assumptions (1) and (2) are valid), the function $|\psi(\mathbf{x}, t)|^2$ will satisfy a similar conservation equation

$$\partial|\psi|^2/\partial t+\operatorname{div}(|\psi|^2\nabla S/m) = \partial|\psi|^2/\partial t+\operatorname{div}(|\psi|^2\mathbf{v})=0. \tag{4}$$

Clearly, then if $P(\mathbf{x})$ is chosen initially equal to $|\psi(\mathbf{x})|^2$, the two will remain equal for all time; and thus, the consistency of assumption (3) is demonstrated. Any failure of assumptions (1) and (2) would, however, also result in the failure of (3), since $P(\mathbf{x})$ would be still conserved, while $|\psi(\mathbf{x})|^2$, in general, would not. Thus, if deviations from (1) and (2) really existed, for example, at the level of 10^{-13} cm, differences between P and $|\psi|^2$ would arise whose cumulative effects would in general be felt even at the atomic level, unless there exist opposing processes which tend continually to re-establish the equality of P and $|\psi|^2$. As we have already pointed out, we shall demonstrate in this paper that randomly distributed collisions furnish just such opposing processes, which in the absence of perturbations from the level of 10^{-13} cm would cause an arbitrary probability density $P(\mathbf{x}, t)$ to decay into $|\psi(\mathbf{x}, t)|^2$ with the passage of time. Clearly this result constitutes an important part of the causal interpretation of the quantum theory, since it shows that the causal interpretation could have an experimental content different from that of the usual interpretation at 10^{-13} cm and still lead to agreement with all experimental data that can now be understood in terms of the usual interpretation.

To show the importance of collisions in determining the probability density, we first note that all experiments in quantum mechanics to date have been concerned with statistics of systems (such as hydrogen atoms) drawn from large aggregates of matter, where they have been interacting with other systems for a very long time. When we do an experiment now, we have no choice but to drawn our samples from such an ensemble, whose members have undergone this lengthy process of collision with other atoms, electromagnetic waves, sound waves, and other disturbances which can alter the physical condition of the members of our ensemble. Now, each collision of, for example, a hydrogen atom with another atom can change the positions of the precisely defined particles located in each atom in a way that depends on the collision parameter and on the initial velocity of approach. Since there is a statistical ensemble of different kinds of collisions, we conclude that even if all the particles in our ensemble initially has the same positions, they would have some kind of distribution after collision.

To study how this probability distribution changes in a collision, we shall find it convenient to define a function $f(\mathbf{x}, t)$ through the equation

$$P(\mathbf{x}, t)=|\psi(\mathbf{x}, t)|^2 f(\mathbf{x}, t). \tag{5}$$

Since $P(\mathbf{x}, t)$ satisfies the conservation equation (3), while $|\psi(\mathbf{x}, t)|^2$ satisfies the similar Eq. (4), we readily find by subtracting (4) from (3) that

$$\partial f/\partial t+\mathbf{v}\cdot\nabla f=0. \tag{6}$$

But the above is just df/dt, the rate of change of f which results from following a particle orbit. We obtain, therefore, $df/dt=0$ and

$$f(\mathbf{x}, t)=f(\mathbf{x}', t'), \tag{7}$$

where $\mathbf{x}'$ is the position of a particle at the time t' which arrives at the position $\mathbf{x}$ at the time t.

Equation (7) is analogous to Liouville's theorem in classical statistical mechanics, with the important difference that in quantum theory, the ratio $f=P/|\psi|^2$ is what remains constant when we follow a moving particle, while in classical statistical mechanics it is the density $\rho(\mathbf{x}, \mathbf{p}, t)$ of points in phase space that remains constant. However, by methods that are very similar to those that can be used in classical statistical mechanics to show that $\rho(\mathbf{x}, \mathbf{p}, t)$ approaches a constant[2] along any

[2] Note that this result is not to be confused with the statement that $\rho(\mathbf{x}, \mathbf{p}, t)$ remains constant when $\mathbf{x}(t)$ and $\mathbf{p}(t)$ are solutions of the equations of motion. For in this case, even though ρ depends on $\mathbf{x}$, $\mathbf{p}$, and t, the dependence is such that there is no net change in ρ when we follow a trajectory of a particle in phase space. However, after many collisions, the functional form of ρ itself

surface of constant energy, we shall show in this paper that $f(\mathbf{x}, t)$ approaches a constant, so that P approaches $|\psi|^2$. Moreover, once P has become equal to $|\psi|^2$, no disturbance satisfying conditions (1) and (2) can possibly produce a difference between these two quantities, because the relation $P=|\psi|^2$ is then maintained for all time by the equations of motion of the particles.

The above results are deduced with the aid of assumptions (1) and (2), which certainly hold in the atomic domain. If, in this domain, perturbations arising at the level of 10^{-13} cm should create discrepancies between P and $|\psi|^2$, we should then expect that these discrepancies would die out in some mean time T, which is determined by the solutions of the equations of motion of the various particles involved and by the rates of collisions of these particles. In the steady state, the difference $P-|\psi|^2$ will be determined by the balance between the mean rate R, at which perturbations surge up from the level of 10^{-13} cm, and the mean rate $(P-|\psi|^2)/T$, at which a perturbation already in existence dies out. This yields

$$P-|\psi|^2=RT. \tag{8}$$

Now in the absence of any specific hypothesis as to what is happening at 10^{-13} cm, it is impossible for us to know anything about R, except that with the limitless number of conceivable hypotheses available any value at all is possible. Thus, if we assume the causal interpretation, we can always regard any experiments which show that P is equal to $|\psi|^2$ as *a posteriori* evidence that R is so small that discrepancies between P and $|\psi|^2$ have not yet been detected. This means that no experiment can possibly show that the usual interpretation must be chosen in preference to the causal interpretation. On the other hand, an experimentally observed discrepancy between P and $|\psi|^2$ would clearly indicate that the usual interpretation was untenable and that a causal interpretation was probably needed.[3]

An actual experimental test of the relation $P=|\psi|^2$ would be impracticably difficult at present, but as shown in paper II, the usual formulas for transition probabilities are, in the causal interpretation, consequencies of the assumption that $P=|\psi|^2$. Hence, if this assumption is not entirely true, we should expect to find discrepancies, probably rather small, between the observed mean rates of transitions and the rates predicted by present theories. We must study these discrepancies under conditions in which ψ is known to a high degree of accuracy. From this point of view, the best transition processes to study would probably be those involving radiation from atoms of hydrogen. However, to reduce the rate at which discrepancies between P and $|\psi|^2$ die out, we should avoid collisions by keeping the atoms under conditions of extreme isolation and avoid the effects of thermal radiation by keeping them at very low temperatures. To increase the rate R at which discrepancies between P and $|\psi|^2$ come up from the level of 10^{-13} cm, we should use atoms whose nuclei are very highly excited. Experimental evidence testing the accuracy of the predictions of the usual interpretation concerning rates of transition is at present rather rough and limited in quantity, and of course, no experiments have as yet been done with a view to finding conditions that are likely to lead to a maximum discrepancy between P and $|\psi|^2$.

changes and approaches a constant along any surface of constant energy. Similarly, from Eq. (7) we see that $f(\mathbf{x}, t)$ is a constant along any particle trajectory regardless of the form of f, but we now assert that after many collisions, the functional form of f changes in such a way that f approaches a constant.

[3] In fact, we can go further and say that after we reach the stage of postulating a particular example of a causal theory, experiments are conceivable (as shown in paper II, Sec. 6) which would permit us to infer the necessity of each element in the hypothesis underlying that particular theory; or in other words, to "observe" or detect that element.

II. PROOF THAT $P(x,t)$ APPROACHES $|\psi(x,t)|^2$

On the basis of the assumption that ψ satisfies Schrödinger's equation and that $\mathbf{v}=\nabla S/m$, we shall now show with the aid of a simple illustrative example that an arditrary probability density $P(\mathbf{x}, t)$ ultimately approaches $|\psi(\mathbf{x}, t)|^2$. This proof will be generalized in a more extensive paper to be published elsewhere.[4]

We choose for our example a hydrogen molecule excited to a doubly degenerate level of energy E_0, in which the component of the angular momentum in the direction of the axis of the molecule is, in the usual interpretation, said to be $\pm\hbar$. In terms of cylindrical polar coordinates, with the z axis along the axis of the molecule, the most general linear combination of these two degenerate eigenfunctions can be written as

$$\psi=\sqrt{2}g(\rho, z)(c_1\cos\phi+c_2\sin\phi)e^{-iE_0t/\hbar}, \tag{9}$$

where c_1 and c_2 are arbitrarily complex coefficients, satisfying the relation

$$|c_1|^2+|c_2|^2=1, \tag{10}$$

and where $g(\rho, z)$ is real and satisfies the normalization condition

$$\int_{-\infty}^{\infty}\int_{0}^{\infty}(g(\rho, z))^2\rho d\rho dz=1. \tag{11}$$

Equation (10) permits us to write

$$c_1=e^{i\alpha_1}\cos\omega;\quad c_2=e^{i\alpha_2}\sin\omega. \tag{12}$$

With these substitutions, we obtain (with $\psi=Re^{iS/\hbar}$,

$$\tan\frac{(S+E_0t)}{\hbar}=\frac{\cos\omega\sin\alpha_i\cos\phi+\sin\omega\sin\alpha_2\sin\phi}{\cos\omega\cos\alpha_1\cos\phi+\sin\omega\cos\alpha_2\sin\phi}. \tag{13}$$

We note that S is a function only of ϕ. This means that the velocity has a component only in the direction

[4] D. Bohm, *Anais da Academia Brasileira de Ciências* (to be published).

of ϕ, given by

$$V_\phi=\rho\frac{d\phi}{dt}=-\frac{1}{\rho}\frac{\partial S}{\partial\phi}=-\frac{1}{\rho}\frac{\cos\omega\sin\omega\sin(\alpha_2-\alpha_1)}{1+\sin\omega\cos\omega\cos(\alpha_2-\alpha_1)\sin2\phi}. \quad (14)$$

Each electron moves in a circle, with a speed that depends on the angle ϕ, as well as on ρ. The solution of (13) is readily seen to be

$$-\tfrac{1}{2}\cos2\phi\cos(\alpha_2-\alpha_1)\sin\omega\cos\omega+\phi-\phi_0 = (t/\rho^2)\cos\omega\sin\omega\sin(\alpha_2-\alpha_1), \quad (15)$$

where ϕ_0 is a constant of integration. Three types of orbits are possible. For $\omega=0, \pm\pi/2, \pm\pi$, we obtain ϕ constant, so that the particle does not move. For ω close to these values (and for ρ large), the particle oscillates on an arc of a circle. For large ratios of $\sin\omega\cos\omega/\rho$, the particle rotates in a complete circle with variable angular velocity.

Let us now consider, for any given ρ, a statistical ensemble of particles all having the same initial wave function, but having an arbitrary initial probability distribution $P_0(\phi)$. Although this distribution will, in general, change with time as a result of particle motions, it is clear that after the particles return to their initial positions, they will have the same separation as they had initially (since each follows a periodic orbit with the same period); so that as long as the molecule is isolated, we shall have only a periodically varying probability distribution. In practice, however, the molecule is not really isolated, since it is subjected to a continual series of collisions with other particles. To study the effects of these collision processes, we must, of course, first solve for the changes in the wave function resulting from the influence of the other particles and then use this solution to calculate the particle motions through the relation $\mathbf{v}=\nabla S/m$.

Now, in our simple illustrative example, it will be adequate to assume an incident particle whose mass M is so great that it can be treated classically (noting, however, that in our more complete article[4] it is shown that the same results will follow if this approximation is not made). In the causal interpretation, a classical approximation for the incident particle means that we can neglect the contribution

$$-(\hbar^2/2M)\nabla_y^2|\Psi(\mathbf{x},\mathbf{y},t)|/|\Psi(\mathbf{x},\mathbf{y},t)|$$

of this particle (whose coordinates are denoted by $\mathbf{y}$) to the quantum potential and that we can approximate the wave equation as

$$i\hbar\frac{\partial\psi}{\partial t}=\frac{1}{2m}\left(\frac{\hbar}{i}\nabla_{\mathbf{x}}-\frac{e}{c}\mathbf{A}(\mathbf{x},\mathbf{y}(t))\right)^2\psi+V(\mathbf{x},\mathbf{y}(t))\psi,$$

where $\mathbf{y}(t)$ is regarded only as a parameter, whose time dependence is obtained by solving the equations of motion, and where $V(\mathbf{x},\mathbf{y})$ is the classical potential energy of interaction between $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{A}(\mathbf{x},\mathbf{y})$ the classical vector potential (due to magnetic interactions).

Let us further restrict ourselves to collisions so distant that perturbation theory may be applied. Now the major changes of the wave function will be due to transitions between degenerate levels, and as will soon be evident, no essential aspect of the problem will be altered if we made the further approximation of neglecting the comparatively unimportant transitions to levels of other energies. The changes in the wave function will then be determined by the three matrix elements

$$V_{11}(t)=2\int\cos\phi g(\rho,z)H_P(\mathbf{x},\nabla_{\mathbf{x}},t)g(\rho,z)\cos\phi d\mathbf{x}, \quad (14a)$$

$$V_{22}(t)=2\int\sin\phi g(\rho,z)H_P(\mathbf{x},\nabla_{\mathbf{x}},t)g(\rho,z)\sin\phi d\mathbf{x}, \quad (14b)$$

$$V_{12}(t)=V_{21}^*(t)=2\int\cos\phi g(\rho,z)H_P(\mathbf{x},\nabla_{\mathbf{x}},t)\times g(\rho,z)\sin\phi d\mathbf{x}, \quad (14c)$$

where $H_P(\mathbf{x},\nabla_{\mathbf{x}},t)$ is the part of the Hamiltonian operator corresponding to the perturbation. These matrix elements define the unitary transformation

$$c_1=\alpha_{11}(t)c_1'+\alpha_{12}(t)c_2', \quad c_2=\alpha_{21}(t)c_1'+\alpha_{22}(t)c_2',$$

where the α_{ij} satisfy the following condition for a unitary matrix

$$|\alpha_{11}|^2+|\alpha_{21}|^2=|\alpha_{12}|^2+|\alpha_{22}|^2=1, \quad \alpha_{11}\alpha_{12}+\alpha_{21}\alpha_{22}=0, \quad (16)$$

and where the c_j' represent the values of the coefficient of the wave function at time t' before collision, while the c_i represent those values at the time t after collision.

The transformation matrix $\alpha_{ij}(t)$ can in principle be obtained by carrying out a series of infinitesimal unitary transformations, with the transformation matrices $(1-iV_{ij}(t)dt/\hbar)$; but because the $V_{ij}(t)$ are all in general different functions of the time, the transformation coefficients $\alpha_{ij}(t)$ cannot in general be expressed in a closed form. Because the dependence of V_{ij} on the time will depend on the orbit of the incident particle, it is clear that the $\alpha_{ij}(t)$ will in general be different for different kinds of collisions. This means that the relation between c_i and c_j' will depend on the collision parameter $\mathbf{h}$ and on the initial velocity of approach $\mathbf{u}$, which quantities determine the orbit of the incident particle in a collision. Thus, we can write

$$c_j'=c_j'(c_i,\mathbf{h},\mathbf{u}).$$

Because the c_i and c_j' satisfy the identity $|c_1|^2+|c_2|^2=|c_1'|^2+|c_2'|^2=1$, which results from the unitary character of the transformation, it is better, however, to work in terms of the three independent parameters introduced in Eq. (12). We obtain

$$\omega'=\omega'(\omega,\alpha_1,\alpha_2,\mathbf{h},\mathbf{u});$$

$$\alpha_1'=\alpha_1'(\omega,\alpha_1,\alpha_2,\mathbf{h},\mathbf{u}); \quad \alpha_2=\alpha_2'(\omega,\alpha_1,\alpha_2,\mathbf{h},\mathbf{u}). \quad (17)$$

Now the various members of our ensemble of hydrogen molecules undergo collisions having a continuously and more or less randomly distributed set of values of $\mathbf{h}$ and $\mathbf{u}$; and as a result even if the c_j' were all initially the same before collisions, they would be distributed statistically after collisions.[5] We are therefore led to define the probability

$$Q(\omega, \alpha_i, \alpha_2, t)d\omega d\alpha_1 d\alpha_2 = Q(c_i, t)d\Omega,$$

where $d\Omega = d\omega d\alpha_1 d\alpha_2$ that the wave-function coefficients lie in the region $d\Omega$. Our problem is then to study how the function Q is changed during a collision. Now in any particular collision, carrying the coefficients from c_j' to c_i, the volume element $d\Omega'$ is carried into some other volume element $d\Omega$. It can then easily be shown that because of the unitary character of the collision,

$$d\Omega' = d\Omega. \tag{18}$$

This follows from the fact that a unitary transformation produces a rotation in the four-dimensional space, whose coordinates are the real and imaginary coefficients of the c_i. Thus, the volume element in the space is not changed, and from this fact we readily obtain Eq. (18). Since the number of systems does not change in a collision, we also have $Q(c_i, t)d\Omega = Q(c_j', t)d\Omega'$ and therefore obtain for the probability density of systems,

$$Q(c_i, t) = Q(c_i', t'). \tag{19}$$

Because the matrix elements $V_{ij}(t)$ vanish as $t \rightarrow \pm\infty$, the coefficients c_i will change only during the short time in which the potential is appreciable. Thus, if t represents any time after the n^{+n} collision and before the $(n+1)^{+n}$, while t' represents any time before the n^{+n} collision and after the $(n-1)^{+n}$, we do not need to consider the explicit time dependence of Q but can write instead

$$Q_{n+1}(c_i) = Q_n(c_j'), \tag{20}$$

where $Q_{n+1}(c_i)$ represents the value of $Q(c_i, t)$ after the n^{+n} collision.

We must now take into account the fact that there is a statistical ensemble of collisions of different types. To do this we define the probability $F(\mathbf{h}, \mathbf{u})d\mathbf{h}d\mathbf{u}$ that a collision occurs with collision parameter in the range $d\mathbf{h}$ and the velocity of approach in the range $d\mathbf{u}$. The next step is to eliminate the components of the collision parameter $\mathbf{h}$, in terms of the variables ω, α_1, α_2, with the aid of Eq. (17). This is permissible only if the Jacobian determinant $J(\partial\omega/\partial h_x, \partial\alpha_1/\partial h_y, \partial\alpha_2/\partial h_z)$, does not vanish identically. The identical vanishing of this determinant would imply that there was a functional relationship between ω, α_1, α_2. But as has already been shown, a three-parameter distribution of collisions (over the variable $\mathbf{h}$ for a given $\mathbf{u}$) will in general produce a three-parameter distribution of the c_i' for a given c_j', so that there can be no functional relationship between ω, α_1, α_2 and the determinant $J(\partial\omega/\partial h_x, \partial\alpha_1/\partial h_y, \partial\alpha_2/\partial h_z)$ cannot vanish identically. We can therefore define the probability,

$$G(c_j', c_i, \mathbf{u}, t, t')d\Omega' d\mathbf{u} = FJ(\partial h_x/\partial\omega, \partial h_y/\partial\alpha, \partial h_z/\partial\alpha_2)d\Omega' d\mathbf{u}, \tag{21a}$$

that the system makes a collision starting in the range $d\Omega' d\mathbf{u}$ and ending up with a given set of wave-function coefficients c_i. Clearly, by definition

$$\int G(c_j', c_i, \mathbf{u}, t')\, d\Omega' d\mathbf{u} = 1. \tag{21b}$$

The precise form of the function G is determined by the distribution function F and by the matrix elements $V_{ij}(t)$ in a very complicated way, but the only property of G that is of interest here is the fact that it is a continuous function of all its variables. This property of continuity follows from the continuity of F, the continuity of the relations (17), and the existence of the determinant $J(\partial h_x/\partial\omega, \partial h_y/\partial\alpha_1, \partial h_z/\partial\alpha_z)$. The continuity of F, however, follows from the assumption of a more or less random distribution of collision parameters and velocities of approach. It is here that we introduce the fundamental statistical element into our treatment.

To calculate the probability density $Q_{n+1}(c_i)$ of wave-function parameters existing after the nth collision, we need only average the contributions to Q_{n+1} coming from the various types of collisions. But each contribution satisfies Eq. (19). Thus, we obtain the following integral equation, which defines the way in which Q_n changes as a result of a collision:

$$Q_{n+1}(c_i) = \int G(c_i, c_j', \mathbf{u})Q_n(c_j')d\Omega' d\mathbf{u}. \tag{22}$$

We are now ready to show that $Q_n(c_j')$ approaches a constant as $n \rightarrow \infty$. First, we note that if $Q_n(c_j')$ is a constant, then by virtue of Eq. (21b), we obtain $Q_{n+1} = Q_n$. Thus, $Q_n =$ constant is a possible equilibrium solution. If $Q_n(c_j')$ is not a constant, then let us denote by A_n its maximum value (as a function of all its variables). This maximum must exist, because by hypothesis, $Q_n(c_j')$ is a continuous function. Let us also denote by B_n the minimum value of $Q_n(c_j')$, and let us restrict ourselves for the time to the case in which there is only one point at which $Q_n(c_j')$ takes on the value A_n (and also only one point at which it takes the value B_n). Then since $Q_n(c_j') < A_n$ for all points except one we

[5] It is important to note here that the range of values of c_i, resulting from a given set of c_j', covers some three-dimensional region (or more precisely a set of finite measure) in the space of ω, α_1, α_2. This can be seen, from the fact that our group of infinitesimal transformations is isomorphic with the group of infinitesimal rotations on a two component spinor. As a result of the continuous distribution over $\mathbf{h}$, we shall have, in general, a continuous distribution of the $V_{ij}(t)$ for a given $\mathbf{u}$; and as a result of the continuous distribution over $\mathbf{u}$, a continuous distribution of the V_{ij} for a given $\mathbf{h}$. Thus, for each value of $\mathbf{h}$ (or of $\mathbf{u}$), we shall obtain a three-parameter distribution over ω, α_1, α_2, for a given ω', α_1', α_2' (or of ω', α_1', α_2' for a given ω, α_1, α_2).

obtain, with the aid of Eq. (21b),

$$Q_{n+1}(c_i)<\int G(c_i, c_j', \mathbf{u})A_n d\Omega' d\mathbf{u}=A_n, \quad (22a)$$

and similarly, since $Q_n(c_j')>B_n$ except for one point, we have

$$Q_{n+1}(c_i)>\int G(c_i, c_j', u)B_n d\Omega' d\mathbf{u}=B_n. \quad (22b)$$

This means that $Q_{n+1}(c_i)$ can nowhere be as large as the maximum of $Q_n(c_j')$ nor as small as the minimum of $Q_n(c_j')$. Hence, the maximum of $Q_{n+1}(c_i)$ must be less than that of Q_n, and its minimum must be greater than that of Q_n. In other words,

$$A_{n+1}<A_n; \quad B_{n+1}>B_n. \quad (23)$$

On the basis of the above results, it seems clear on intuitive grounds that $Q_n(c_i)$ must approach a constant as $n\to\infty$. To prove that this happens, we shall show that the hypothesis that it does not approach a constant leads to a contradiction. We let $Q_\infty(c_i)$ represent the limiting form of $Q_n(c_i)$. We can then write $Q_n(c_i)=Q_\infty(c_i)+\epsilon_n(c_i)$, where $\epsilon_n(c_i)$ is a term that approaches zero uniformly as $n\to\infty$. Let us now evaluate $Q_n(c_i^m)-Q_{n+1}(c_i^m)$, where c_i^m is the point at which $Q_\infty(c_i)$ has its maximum value A_∞. We have

$$Q_n(c_i^m)-Q_{n+1}(c_i^m)=A_\infty-Q_{n+1}(c_i^m)+\epsilon_n(c_i^m),$$

and

$$A_\infty-Q_{n+1}(c_i^m)=\int G(c_i^m, c_j', \mathbf{u}) \times(A_\infty-Q_\infty(c_j')+\epsilon_n(c_j'))d\Omega' d\mathbf{u}.$$

But if $Q_\infty(c_j')$ is not a constant function of its arguments, then in the region in which G is appreciable, $A_\infty-Q_\infty(c_j')$ will be some finite number which does not approach zero as $n\to\infty$. We conclude then that $Q_n(c_i^m)-Q_{n+1}(c_i^m)$ also does not approach zero as $n\to\infty$. This contradicts the hypothesis that $Q_n(c_i)$ approaches $Q_\infty(c_i)$ as a limit, since such an approach requires that $Q_{n+1}(c_i)-Q_n(c_i)$ approach zero for all c_i. We conclude then that $Q_n(c_i)$ must approach a constant as $n\to\infty$.

We shall now remove the limitation that $Q_n(c_j')$ takes on its maximum (or minimum) value at only one point. First, we note that if $Q_n(c_j')$ takes on the value A_n (or B_n) on a set of isolated points, curves or surfaces, of dimensionality lower than that of the space of the ω, α_1, α_2 (or more generally, on a set of measure zero in this space), then it is obvious that the reasoning goes through precisely the same lines as before. If, however, $Q_n(c_j')$ should happen to take on the value A_n in a three-dimensional domain (or more generally in a region of finite measure in the space of ω, α_1, α_2) and if this region is bigger than the domain in which $G(c_i, c_j', \mathbf{u})$ is different from zero, the problem is a little more complicated, because one then obtains the result that $A_{n+1}=A_n$. However, if we evaluate $Q_{n+1}(c_i)$ near the edge of such a domain, we obtain contributions to the integral from regions in which $Q_n(c_j')<A_n$, so that at such points, $Q_{n+1}(c_i)<A_n$. Thus, in each collision, the special domain in which $Q_n(c_j')=A_n$ is narrowed until there is left only an isolated maximum point, and the problem is then reduced to the case originally considered.

The significance of our result is that after many collisions, the probability density for any set of wave-function coefficients, as defined by the ω, α_1, α_2, will be uniform. This means that each coefficient has a random phase and that there is a uniform probability for all values of ω (ω defines the absolute magnitude of the wave-function coefficients $\cos\omega$ and $\sin\omega$, of c_1 and c_2, respectively). But this is exactly what would be obtained from the usual quantum-statistical mechanics for the case of an ensemble of systems, each having two degenerate levels. This result is not surprising, since the causal interpretation treats the wave function in exactly the same way as does the usual interpretation.

It is important to note that the continuity of the function $G(c_i, c_j', \mathbf{u})$ played an essential role in the proof that $Q_n(c_i)$ approaches a constant as $n\to\infty$. Thus, if G had been a function that was zero everywhere except at certain isolated points, we could not have shown that $A_{n+1}<A_n$. For in this case, if the function $Q_n(c_j')$ had taken on its maximum value at precisely those points where G did not vanish, we should have obtained $A_{n+1}=A_n$. But the continuity of G guarantees that the integral expressing $Q_{n+1}(c_i)$ in terms of $Q_n(c_j')$ must obtain contributions from regions in which $Q_n(c_j')$ is less than its maximum, so that $A_{n+1}<A_n$. As we have seen, the continuity of G originates in the assumption of a more or less random distribution of types of collision. Thus, the approach of $Q_n(c_i)$ to its equilibrium value is based on the random character of the collision processes.

Let us now consider the particle motions. In general, these are rather difficult to solve for; but as in the case of the wave-function coefficients we are not interested in the details, but only in the fact that a statistical ensemble of collisions will lead to a corresponding statistical ensemble of changes in particle positions. By integrating the relation $\mathbf{V}=\nabla S/m$, we see that the initial position $\mathbf{x}'$ of a particle is a function of the final position $\mathbf{x}$, the final wave-function coefficients c_i, the collision parameter $\mathbf{h}$, and the initial velocity of approach $\mathbf{u}$. But as shown in Eq. (17), we can eliminate $\mathbf{h}$ in terms of c_i and c_j'. We then get

$$\mathbf{x}'=\mathbf{x}'(c_i, c_j', \mathbf{u}, \mathbf{x}, t, t'). \quad (24)$$

We must explicitly retain t and t' here, because the particles are in general moving both before and after the collision.[6] However, it will be convenient to define

[6] This can be seen from Eq. (14). Since the phase difference $\alpha_2-\alpha_1$ appearing in this equation is not in general zero either before or after collision, and since $\cos\omega\sin\omega$ is also not in general zero, v_φ will in general differ from zero except in the very special cases mentioned above, which correspond to isolated possibilities of probability zero.

t' in such a way that it represents what may be called the "beginning" of the collision. This could be done, for example, by choosing it as the time when the distance of the incident particle is the smallest value d_0 for which the interactions can be neglected.

Now, because the relation between $\mathbf{x}'$ and $\mathbf{x}$ depends on $\mathbf{u}$, which is distributed statistically, we conclude that even if those particles having the same wave-function coefficients before and after collision had the same initial positions $\mathbf{x}'$, they would have a statistical distribution of positions $\mathbf{x}$ after collision.[7] This leads us to define the probability $P(c_i, \mathbf{x}, t)d\mathbf{x}$ that if a hydrogen molecule has wave-function coefficients c_i, its electron is located in the region $d\mathbf{x}$. In accordance with Eq. (5), we can write

$$P(c_i, \mathbf{x}, t) = |\psi(c_i, \mathbf{x}, t)|^2 f(c_i, \mathbf{x}, t), \tag{25}$$

where by definition,

$$\int P(c_i, \mathbf{x}, t)d\mathbf{x} = \int |\psi(c_i, \mathbf{x}, t)|^2 d\mathbf{x} = 1. \tag{26}$$

To obtain the probability density for the entire ensemble, we must multiply the above by the probability density $Q(c_i)$ that a molecule has the wave-function coefficients c_i. Using Eq. (7), we also have $f(c_i, \mathbf{x}, t) = f(c_j', \mathbf{x}', t')$; and using Eq. (19), we have $Q(c_i, t) = Q(c_j', t')$. Thus, we can write for the probability density in the space of $\mathbf{x}$, ω, α_1 and α_2,

$$Q(c_i, t)P(c_i, \mathbf{x}, t) = |\psi(c_i, \mathbf{x}, t)|^2 Q(c_j', t')f(c_j', \mathbf{x}', t'). \tag{27}$$

To calculate the probability density after the nth collision, we must average the contribution to $Q(c_i, t)P(c_i, \mathbf{x}, t)$ coming from the various types of collisions. Each contribution satisfies Eq. (27). But each contribution must be weighted with the proper weighting function. To obtain the weighting function, we begin with Eq. (21), which gives the probability of a collision with velocity of approach in the range $d\mathbf{u}$, that carries the wave-function coefficients from the region $d\Omega'$ to the definite values c_i. We must now eliminate $\mathbf{u}$ in terms of $\mathbf{x}$ with the aid of Eq. (24). However, we must note here that in our example of two degenerate levels, only ϕ will be changed in a collision,[8] and not ρ and z. Our simple example therefore serves only to illustrate the approach to equilibrium of the distribution in ϕ; but it is easily shown[4] that a more complex example would also demonstrate a similar approach to statistical equilibrium in ρ and z. In this example, we shall hereafter restrict ourselves to considering P and f as functions only of ϕ.

Now, even when c_i and c_j' are fixed, a continuous variation in any component of $\mathbf{u}$, say u_z, will in general produce a corresponding continuous variation[9] of ϕ'. In other words, if we solve for ϕ' with Eq. (24), we shall find that $\partial\phi'/\partial u_z$ is a continuous function, not identically zero. Thus, we can eliminate u_z in Eq. (21), expressing it in terms of ϕ', and obtain the probability,

$$K(c_i, c_j'; \phi, \phi'; u_x, u_y; t, t') = G(\partial u_z/\partial\phi')d\phi' d\Omega' du_x du_y,$$

that a collision occurs with approach velocity components in the range $du_x du_y$ and which carries the wave-function coefficients from $d\Omega'$ to c_i and the angle of the particle from $d\phi'$ to ϕ. Note that by definition

$$\int K d\Omega' d\phi' du_x du_y = 1. \tag{28}$$

The function $K d\Omega' d\phi' du_x du_y$ is clearly the proper weighting function to use with Eq. (27). Denoting by f_{n+1} the value of f after the nth collision, we then obtain

$$Q_{n+1}(c_i)P_{n+1}(c_i, \phi, t) = |\psi(c_i, \phi, t)|^2 \int K(c_i, c_j'; \phi, \phi'; u_x, u_y, t, t') \times Q_n(c_j')f_n(c_j', \phi', t')d\Omega' du_x du_y d\phi'. \tag{29}$$

But the above must also be averaged over the times t' at which a collision begin, which are distributed at random. In doing this, it is convenient to average over a time T long compared to the time necessary to complete a collision but short compared with the time between collisions (such a case can always be obtained by making the pressure low enough). We then obtain [with the aid of (25)] an integral equation defining how $Q_n f_n$ is changed in these systems undergoing collision between t and $t-T$:

$$Q_{n+1}(c_i)f_{n+1}(c_i, \phi, t) = \int_{t-T}^{t} \frac{dt}{T} \int K Q_n(c_j') \times f_n(c_j'\phi', t')d\Omega' d\phi' du_x du_y. \tag{30}$$

From here on, the proof that $Q_n f_n$ approaches a constant as $n \to \infty$ is much the same as was the proof that Q_n approaches a constant. We let C_n be the maximum of $Q_n f_n$ and D_n its minimum. Then using (28), we readily show that

$$Q_{n+1}f_{n+1} < C_n; \quad Q_{n+1}f_{n+1} > D_n. \tag{31}$$

Note, however, that to obtain this result, we must, as

[7] This result is proved in detail in the more extensive paper mentioned in reference 4. However, it is very plausible, since different collision parameters will produce different changes of the wave-function coefficients throughout the collision, and therefore different velocities $\mathbf{v} = \nabla S/m$, so that different distances would be covered by the particles in the resulting motions.

[8] This can be seen from Eq. (14) and (14a, b, c). In the present approximation of neglecting transitions to nondegenerate levels, the phase S, and therefore the velocity $\mathbf{v} = \nabla S/m$, can depend only on ϕ; but if more terms were included in the expansion (9) of the wave function, S would clearly depend on ρ and z also. For a more detailed treatment of this point, see the paper mentioned in reference 4.

[9] As shown in paper II, Sec. 6, if the usual theory should fail in any domain, such as that associated with 10^{-13} cm, the positions of these particles could be observed with unlimited precision, so that it would be possible, at least in principle, to predict the precise time of decay of an individual nucleus.

in obtaining the similar result for Q_n, use the fact that K is a continuous function of all its variables. It is here that we bring in the basic hypothesis of a statistically distributed series of kinds of collisions.

Thus, we can deduce that $Q_n f_n$ approaches a constant. Since Q_n approaches a constant, we deduce that f_n also approaches a constant, which must be unity, since by (26) both P and $|\psi|^2$ are by definition normalized. This means that $P(c_i, \phi, t) = |\psi(c_i, \phi, t)|^2$, or that if we fix our attention on those cases in which the wave-function coefficients are c_i, the probability density is just $|\psi(c_i, \phi, t)|^2$.

Finally, let us note that no other solution of the conservation equation could have taken the place of $|\psi(c_i, \phi, t)|^2$ in our proof that after many collisions, $P(c_i, \phi, t) \rightarrow |\psi(c_i, \phi, t)|^2$. Although Eq. (7) would have followed from Eq. (5) if we had replaced $|\psi(\mathbf{x}, t)|^2$ by an arbitrary solution $H(\mathbf{x}, t)$ of the conservation equation, the step from Eq. (27) to Eq. (29) would not then have been possible. For in this step, we used the property that the value of $|\psi(c_i, \phi, t)|^2$ depends *only* on the final coefficients c_i of the wave function and not on the initial coefficients c_j'. This property makes it possible for us to take $|\psi(c_i, \phi, t)|^2$ out of the integral as a common factor, independent of the initial wave-functions coefficients c_j'. With any other solution of the conservation equation, the final value of the function would have depended on the initial coefficients c_j', as well as on the c_i, so that the function could not have been taken out of the integral and the proof would not have gone through.

III. SUMMARY AND CONCLUSIONS

We shall summarize the results of this paper in terms of the simple example of the α-decay of two uranium nuclei having the same wave function. Now one of these nuclei may disintegrate tomorrow and the other in two billion years. The usual interpretation of the quantum theory states, however, that *today* there can be no physically describable difference between these nuclei, since they both have the same wave function and since the wave function is said to determine all physically significant properties of a system. But the most elementary scientific procedure would suggest that if two objects are observed to act differently, this should be regarded as *a posteriori* evidence that there must in fact be some physical difference between them. Indeed, in the causal interpretation, the difference in the two uranium nuclei is explained very simply in terms of the assumption that each nucleus has in it a set of particles with precisely defined positions, which determine in principle exactly when that nucleus is going to decay (see paper I, Sec. 8). The difference in times of disintegration is then ascribed to the differences in positions of the various particles in the two nuclei. In fact, if we consider a statistical ensemble of such nuclei, all having the same wave function, then as has been shown in this paper, the particles in different nuclei will be in different positions as a result of collisions suffered by these nuclei over the past few billion years; and these collisions will have produced a probability density of $P = |\psi|^2$ that the particles in a particular nucleus take a given set of positions. Thus, we are able, as shown in paper I, Sec. 8, to explain the statistical ensemble in decay times of a large number of nuclei.

In the usual interpretation, however, the role of statistics is very different, for it is said that the precise future behavior of an individual system is completely arbitrary, in the sense that it cannot be related to any physically definable properties of that system or of anything else existing in the world today. Only the probability of decay in a statistical ensemble of systems is said to be determined by physical factors now in existence. Nevertheless, it is admitted that the behavior of an individual system such as a uranium nucleus can be physically significant, since, for example, if this nucleus disintegrates, the resulting particle can activate a Geiger counter, which can initiate a large scale process, such as the setting off of a bomb. In this situation, the question of when a particular nucleus decays clearly has physical significance, since it determines, for example, whether the bomb will explode tomorrow or in two billion years. In accordance with the postulate that only the behavior of a statistical ensemble of uranium nuclei can be determined by physical factors now in existence, however, the usual interpretation states that the time at which a particular bomb goes off is completely arbitrary, from a physical point of view.

The arbitrariness of the usual interpretation in the description of the behavior of an individual system is closely related to the assumption, already stated, that the wave function determines all physically significant properties of that system. Now, in the case of the uranium nucleus, the wave function takes the form of a packet initially entirely within the nucleus, which gradually "leaks" through the barrier and thereafter rapidly spreads without limit in all directions. Clearly, although this wave function is supposed to describe *all* physically significant properties of the system, it cannot explain the fact that each α-particle is actually detected in a comparatively small region of space and at a fairly well-defined instant of time. The usual interpretation states that this phenomenon must simply be accepted as an event that somehow manages to occur but in a way that is as a matter of principle forever beyond the possibility of a simultaneous and detailed "space-time and causal description." Indeed, even to ask for such a description is said to be a meaningless question within the framework of the usual interpretation of the quantum theory. In the causal interpretation, however, the postulated particles with precisely defined positions explain in a natural way why an α-particle can be detected at a fairly definite place and time, on the basis of the simple assumption that the particle existed all the time and just moved from its original location

to the place where it was finally found. Thus, even though we cannot yet[9] observe the precise locations of our postulated particles, they already perform a real function in the theory, namely, to explain certain properties of *individual* systems which are said in the usual interpretation to be just empirically given and forever unexplainable.

The postulation of particles with precisely defined (but not at present precisely measurable) positions not only makes possible a more connected description of the behavior of an individual system than is possible in the usual interpretation but also serves to increase the degree of unity in the treatment of the statistical aspects of the theory. Thus, in the usual interpretation, two completely different kinds of statistics are needed. First, there is the ordinary statistical mechanics, which treats of the distribution of systems among the quantum states, resulting from various chaotic factors such as collisions. The need for this type of statistics could in principle be avoided by means of more accurate measurements which would supply more detailed information about the quantum state, but in systems of appreciable complexity, such measurements would be impracticably difficult. Secondly, however, there is the fundamental and irreducible probability distribution, $P(\mathbf{x})=|\psi(\mathbf{x})|^2$, or more generally, $P_a=|C_a|^2$, where P_a is the probability that in the measurement of an arbitrary observable A we shall obtain the eigenvalue a, corresponding to the eigenfunction $\psi_0(\mathbf{x})$, and where $\psi=\sum_a C_a\psi_a(\mathbf{x})$. The need for this type of statistics cannot even in principle be avoided by means of better measurements, nor can it be explained in terms of the effects of random collision processes. The usual interpetation simply postulates the above probability distribution as a basic and not further analyzable property of matter, in addition to the more familiar type of probability that applies in the statistical distribution of systems among the quantum states. On the other hand, the causal interpretation requires only one kind of probability. For as we have seen, we can deduce the probability distribution $P(\mathbf{x})=|\psi(\mathbf{x})|^2$ as a consequence of the same random collision processes that give rise to the statistical distributions among the quantum states. Moreover, as shown in paper II, this result is sufficient to prove the more general result that the probability of obtaining the eigenvalue a in the measurement of an arbitrary observable A is $P_a=|C_a|^2$. In this way, the causal interpretation avoids the need for introducing *ad hoc* a completely new type of probability distribution, which does not represent incompleteness of information and which is not even in principle explainable in terms of random processes, such as collisions.

As the situation stands now, the causal interpretation and the usual interpretation each introduce one hypothesis that has not yet been proved experimentally, the causal interpretation assuming the existence of particles with precisely defined positions and the usual interpretation assuming that $P(\mathbf{x})=|\psi(\mathbf{x})|^2$ (or more generally, $P_a=|C_a|^2$).[10] The causal interpretation already has, however, the advantage of providing a more unified description of nature than is possible in the usual interpretation, one involving a smaller number of hypotheses that must simply be accepted as empirical facts without further explanation. For by making *one* postulate, that of the existence of particles with precisely defined positions, the causal interpretation explains two general properties of matter which must simply be postulated in the usual interpretation, i.e., the appearance of an individual particle in a fairly definite position and at a fairly well-defined instant of time after the wave function has spread over a wide region of space, and the probability distribution of these particles in a statistical ensemble of systems having the same wave function. For this reason, it would seem that the assumption of particles with precisely defined positions is likely to be on the right track, at least in its essential features, even if all of the details of the theory thus far suggested may perhaps not appear in a better theory of the phenomena associated with distances of the order of 10^{-13} cm.

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[10] As shown in Sec. I of the present paper, the assumption that $P(\mathbf{x})=|\psi(\mathbf{x})|^2$ has not yet been verified experimentally, nor is it likely that experiments of the requisite precision will be possible in the near future.